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# Empirical Studies in a Multivariate Non-stationary, Nonparametric Regression Model for Financial Returns

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## Abstract

In this paper we analyze a multivariate non-stationary regression model empirically. With the knowledge about unconditional heteroscedasticity of financial returns, based on univariate studies and a congruent paradigm in Gürtler and Rauh (2009), we test for a time-varying covariance structure firstly. Based on these results, a central component of our non-stationary model is a kernel regression for pairwise covariances and the covariance matrix. Residual terms are fitted with an asymmetric Pearson type VII distribution. In an extensive study we estimate the linear dependence of a broad portfolio of equities and fixed income securities (including credit and currency risks) and fit the whole approach to provide distributional forecasts. Our evaluations verify a reasonable approximation and a satisfactory forecasting quality with an outperformance against a traditional risk model.

*JEL classification: C14, C5*

*Keywords: heteroscedasticity, non-stationarity, nonparametric regression, volatility, covariance matrix, innovation modeling, asymmetric heavy-tails, multivariate distributional forecast, empirical studies*

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## 1 Introduction: Motivation and modeling framework

A plenty of studies and models focus on the multivariate, coherent dynamics of financial returns. The classical approach is the random walk hypothesis for log-returns  $X_t = \ln P_t - \ln P_{t-1}$  of prices  $\{P_t\}_{t=0,\dots,n}$ , that are assumed to be independent identically distributed (iid) Gaussian increments (see Bachelier (1900) or Samuelson (1973)). Common practical applications are amongst others the Markowitz (1952) portfolio theory and the related delta-normal model (compare Jorion (2006)) for measuring portfolio risk. The latter approach, also called parametric Value at Risk (VaR) since it works with a certain quantile of the return distribution, matches asset exposures with iid normal risk factors and aggregates with their correlation matrix. The random walk hypothesis has been denied statistically for the majority of financial instruments or return series long since (compare inter alia Fama (1965) rejecting Gaussian features, Lo and MacKinlay (1988) rejecting moreover a (stationary) independence).<sup>1</sup> Yet under discussion remains the long range dependence (LRD) effect (see Taylor (1986)): correlograms  $\hat{\rho}_X(h)$  of (stationary) time series  $\{X_t\}_{t=1,\dots,n}$  exhibit little serial correlation of daily returns, but their absolutes  $|X_t|$  are significantly positive correlated, leading to a slowly declining sample autocorrelation function (SACF) over a larger number of lags. One might draw the conclusion of serial non-linear dependence (= LRD), however Mikosch and Starica (2004) and Granger and Starica (2005) derive theoretically and empirically that the SACF shape arises mainly from non-stationarities in the data due to structural breaks of the unconditioned variance.

As an alternative, nonlinear methods of time series analysis were introduced by Engle (1982) and Bollerslev (1986) as autoregressive conditional heteroscedastic (ARCH) processes, that focus on serial dependence and conditional time-varying volatility. The general form is

$$X_t = \mu_t + \varsigma_t \varepsilon_t, \quad t = 1, \dots, n, \quad (1)$$

where  $\{\varepsilon_t\}_t$  is an iid sequence with  $E\varepsilon_1 = 0$  and  $\text{Var}\varepsilon_1 = 1$ ,  $\{\mu_t\}_t$  and  $\{\varsigma_t\}_t$  are stochastic processes dependent on past information. In Engle's ARCH( $p$ ) model conditional volatility dynamics  $\varsigma_t$  are linear regressed over past (squared) returns, whereas Bollerslev's GARCH( $p, q$ ) extension includes additionally past variances into the parametric regression. For the purpose of parameter estimation processes of the ARCH-family are defined to be stationary, but finding conditions for the existence and uniqueness of a stationary solution is nontrivial (see Bougerol and Picard (1992)). Features of a heavy-tailedness and asymmetry are imposed either on the distribution of innovations  $\varepsilon_t$  or in enhanced definitions of the volatility dynamics as in the exponential GARCH (EGARCH) of Nelson (1991) or the asymmetric power GARCH (AGARCH) of Ding et al. (1993). More and more sophisticated variants of the volatility processes, that are specific for singular financial instruments or situations but not universal, were developed; Bollerslev et al. (1994) give a statistical overview of model versions. To model the comovements of financial returns multivariate GARCH (MGARCH) models were generalized, as the VEC-GARCH model of Bollerslev et al. (1988) or the constant conditional correlation model (CCC-GARCH) of Bollerslev (1990). A review of multivariate GARCH models is provided by Bauwens et al. (2006) or Silvennoinen and Teräsvirta (2008). As the exist-

<sup>1</sup>Regarding the 30 benchmark series in this article, all aspects of the iid Gaussian return assumption are investigated in a comprehensive survey of Gürtler and Rauh (2009), rejecting the normality, the serial identity and the independence for daily samples, but giving strong support for non-stationarity and heteroscedasticity.

ing parametric MGARCH models could not capture the non-linearity and non-normality of widely observed financial data, Long and Ullah (2005) suggest nonparametric and semiparametric extensions for estimating the conditional covariance matrix.

General criticism on ARCH-type models is based in their parametrization as stationary processes (i.e. with a fixed unconditional variance) and their description of the dependence structure of second moments (modeling the LRD). Another inconsistency of long-term GARCH(1, 1) implementations is that typically the sum of estimated parameters is nearly one, leading to an IGARCH(1, 1) model. But this implies an infinite variance of the observed random variables, which contradicts to the results of a direct tail analysis indicating that daily returns have a finite second moment (see De Haan et al. (1994)). Mikosch and Starica (2004) prove again that the IGARCH effect may be generated by non-stationarities via shifts in the unconditional variance of the return series. Drees and Starica (2002) conclude from the need of an increasing complexity for volatility modeling that a simple endogenous specification does not exist and change the working hypothesis: in their univariate approach the volatility is supposed to be exogenous to the return process, and the evolution of market prices is interpreted as a manifestation of complex market conditions. In the return model (1) the volatility term is replaced by an unconditional variable  $\sigma(t)$ , where the process  $\{\sigma(t)^2\}_t$  is modeled as a discretization of a smooth, deterministic function of time via nonparametric kernel regression. A special focus is set on an accurate description of the innovations  $\{\varepsilon_t\}_t$  by fitting an asymmetric version of the Pearson type VII distribution.<sup>2</sup> Herzel et al. (2005) extend these ideas to the multivariate, non-stationary regression framework.<sup>3</sup>

In this article we will directly refer to the approach of Herzel et al. (2005) and the related theory following Gürtler et al. (2009). The vectors of financial returns are assumed to have a smoothly, time-varying unconditional covariance matrix, that is modeled exogenous deterministic via classical nonparametric regression with equidistant design points. The standardized residuals are modeled parametrically, allowing for asymmetry and heavy tails as positive and negative innovations are separately Pearson type VII distributed. This leads to a multiplicative approach, with a constant mean return  $\boldsymbol{\mu}$  added, for a non-stationary sequence  $\{\mathbf{X}_t\}_t$  of independent random vectors (rvecs):

$$\mathbf{X}_t = \boldsymbol{\mu} + \mathbf{S}_t \boldsymbol{\varepsilon}_t, \quad t = 1, \dots, n, \quad (2)$$

$\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n$  iid rvecs with mutually independent coordinates,

$$E\varepsilon_{k,1} = 0, \text{Var}\varepsilon_{k,1} = 1, \quad \forall k = 1, \dots, d,$$

$\mathbf{S}_t : [0, n] \rightarrow \mathbb{R}^{d \times d}$  is an invertible matrix and a smooth function of time.

The aim is to estimate the multivariate return dynamics only by dint of recent returns and to build up short-term forecasts of future return distributions in a similar economic environment. Regarding the included

<sup>2</sup>Drees and Starica (2002) provide one example of a 12-year S&P 500 return series where their non-stationary model fits the data adequately and gets better short-term forecasts on the return distribution than conventional GARCH models. Gürtler and Rauh (2009) and Gürtler and Rauh (2012) investigate the properties of the univariate implementation for simulated prices process and a multitude of financial times series, observing satisfying model approximations and good forecasting abilities, that outperform the delta-normal-model and famous ARCH-type models.

<sup>3</sup>Another univariate extension with a time-varying expected return was developed by Mikosch and Starica (2003).

return information we have to distinguish between a two-sided (symmetrical) implementation and a one-sided (historical) implementation, that is applied for forecasting return dynamics. As introduced in Grtler et al. (2009) three steps have to be arranged to fit the regression-model to a financial time series.

**1. Centering returns:** The demeaned return series  $\{\mathbf{R}_t\}_t$  and  $\{\tilde{\mathbf{R}}_t\}_t$  are defined as:

(I) Two-sided centered return:

$$\mathbf{R}_t := \mathbf{X}_t - \bar{\mathbf{X}}_n, \quad (3)$$

(II) One-sided centered return:

$$\tilde{\mathbf{R}}_t := \mathbf{X}_t - \bar{\mathbf{X}}_{t-1}, \quad (4)$$

with column vectors centered component-wise by the empirical mean,  $\bar{X}_{k,m} = \frac{1}{m} \sum_{t=1}^m X_{k,t}$  for  $m \in \mathbb{N}$  and  $k = 1, \dots, d$ .<sup>4</sup> If the estimation error of  $\bar{\mathbf{X}}_t$  for  $\boldsymbol{\mu}$  was neglected,  $\{\mathbf{R}_t \mathbf{R}'_t\}_t$  (or  $\{\tilde{\mathbf{R}}_t \tilde{\mathbf{R}}'_t\}_t$  respectively) is an independent sequence of matrices with pointwise expectations  $E(\mathbf{R}_t \mathbf{R}'_t) = \mathbf{S}_t \mathbf{S}'_t =: \boldsymbol{\Sigma}^2(t)$ , a smooth function of time. Hence, variances and covariances in matrices  $\{\boldsymbol{\Sigma}^2(t)\}_t$  may be estimated by kernel regression on equidistant design points  $t = \{1, \dots, n\}$  of the squared (centered) return vectors.

**2. Estimating covariances:** With the tools of classical nonparametric regression, multivariate Nadaraya-Watson estimators (MNWE) can be derived from a local polynomial regression method (local constant regression) on  $\{\mathbf{R}_t \mathbf{R}'_t\}_t$ , with data localized by symmetric kernel functions  $K_h$  on a compact support with bandwidth  $h > 0$ , and a method of least squares. We distinguish two versions of estimation:

(I) Two-sided MNWE (smoother):

$$\hat{\boldsymbol{\Sigma}}^2(t) = \frac{\sum_{i=1}^n K_h(i-t) \mathbf{R}_i \mathbf{R}'_i}{\sum_{i=1}^n K_h(i-t)}, \quad (5)$$

(II) One-sided MNWE (filter):

$$\hat{\boldsymbol{\Sigma}}^2_{(1)}(t) = \frac{\sum_{i=1}^t K_h(i-t) \tilde{\mathbf{R}}_i \tilde{\mathbf{R}}'_i}{\sum_{i=1}^t K_h(i-t)}, \quad (6)$$

with  $K_h(\cdot) = \frac{1}{h} K\left(\frac{\cdot}{h}\right)$ . Therein  $h \in \mathbb{N}$  is an appropriate bandwidth (or in a special conception a matrix of bandwidths  $\mathbf{h} \in \mathbb{N}^{d \times d}$ ) and  $K$  is an appropriate kernel, both defined in section 3.

**3. Fitting innovations:** The realized innovations  $\varepsilon_{k,t}$  are component-wise estimated by dint of demeaned returns  $R_{k,t}$  and estimated volatilities  $\hat{\sigma}_k(t) = \left(\hat{\boldsymbol{\Sigma}}_{k,k}(t)\right)_{i,j=1,\dots,d}$  (the  $k$ -th diagonal element of the square root of the estimate  $\hat{\boldsymbol{\Sigma}}^2(t)$  for  $\mathbf{S}_t \mathbf{S}'_t$ ) or  $\hat{\sigma}_{(1)k}(t) = \left(\hat{\boldsymbol{\Sigma}}^2_{(1)k,k}(t)\right)_{i,j}$ , respectively:

(I) Two-sided innovation estimate:

$$\hat{\varepsilon}_{k,t} = \frac{R_{k,t}}{\hat{\sigma}_k(t)}, \quad t = 1, \dots, n, \quad (7)$$

(II) One-sided innovation estimate:

$$\hat{\varepsilon}_{k,t} = \frac{\tilde{R}_{k,t}}{\hat{\sigma}_{(1)k}(t)}, \quad t = 1, \dots, n. \quad (8)$$

<sup>4</sup>  $\bar{X}_{k,0} = \mu_k$  by definition, with  $\mu_k$  a technical, unspecified constant of the  $k$ -th return series.

Due to their independence it is sufficient to specify the distributions of  $\hat{\varepsilon}_{k,t}$ ,  $k = 1, \dots, d$  univariate. The Pearson type VII distribution was found to be a flexible and parsimonious family of (heavy-tailed) distributions. It has the following one-sided density with shape parameter  $m$  and scale parameter  $c$ :

$$f_{m,c}^{\text{VII}(1)}(x) = \frac{2\Gamma(m)}{c\Gamma(m - \frac{1}{2})\pi^{1/2}} \left(1 + \left(\frac{x}{c}\right)^2\right)^{-m} I_{[0,\infty)}(x). \quad (9)$$

With respect to asymmetries it is fitted separately to nonnegative innovations  $\{\hat{\varepsilon}_{k,t} \mid \hat{\varepsilon}_{k,t} \geq 0\}_t =: \varepsilon_+$  and absolute values of negative innovations  $\{-\hat{\varepsilon}_{k,t} \mid \hat{\varepsilon}_{k,t} < 0\}_t =: \varepsilon_-$ . It may be assumed that the median of innovations is 0 and the one-sided Pearson type VII densities  $f_{m_+,c_+}^{\text{VII}(1)}$  and  $f_{m_-,c_-}^{\text{VII}(1)}$  are combined as

$$f_{m_+,c_+,m_-,c_-}^{\text{VII}}(x) = \frac{1}{2} \left( f_{m_-,c_-}^{\text{VII}(1)}(-x)I_{(-\infty,0)}(x) + f_{m_+,c_+}^{\text{VII}(1)}(x)I_{[0,\infty)}(x) \right), \quad (10)$$

which is referred as asymmetric Pearson type VII distribution. The corresponding parameters  $(m_+, c_+)$  and  $(m_-, c_-)$  are estimated with a method of moments, adopted from Gurtler et al. (2009). For each tail

$$m. = \frac{5\beta_2 - 9}{2\beta_2 - 6}, \quad c. = \sqrt{\frac{2\beta_2\mu_2}{\beta_2 - 3}}, \quad (11)$$

where  $\mu_2(\varepsilon.) = E\varepsilon^2$  and  $\beta_2(\varepsilon.) = \frac{E(\varepsilon. - E\varepsilon.)^4}{(E(\varepsilon. - E\varepsilon.)^2)^2}$  (kurtosis), with  $\beta_2 > 3$  being required. Empirical estimates for  $\mu_2$  and  $\beta_2$  are inserted regarding subsamples  $\varepsilon_+$  and  $\varepsilon_-$ .<sup>5</sup> Moreover, the symmetric Pearson VII distribution equates to a scaled Student- $t$  distribution with  $2m - 1$  df. Amongst others, it holds for the innovation quantiles that  $u_{m,c;\alpha} = \frac{c}{\sqrt{2m-1}} t_{2m-1;\alpha}$  (with the former being the Pearson VII and the latter the Student  $\alpha$ -quantile). This enables a factor-based approach for the task of VaR calculation, we apply in section 4.

Our article is thought as the empirical application of the multivariate non-stationary approach including a broad study of the dynamics of real financial time series. Evidence is provided that return vectors are heteroscedastic with time-varying correlation structures, which is exemplified by a 30-dimensional benchmark universe of equity indices, interest rates, credit spreads and exchange rates. We extend the existing literature on the nonparametric regression approach (2) as we analyze two alternatives of smoothing covariance matrix estimates beyond the introduction of two- and one-sided MNWEs (5) and (6): advantages and disadvantages of a wholistic vs. an individual smoothing approach are investigated. We are the first to adopt the whole paradigm, that incorporates also asymmetric and heavy-tailed vector components, to such a large sample of interrelated returns, and we address the potentials of automatization. Moreover, we aim at developing an analytical and a simulative access to estimate portfolio VaR. Forecasts of multivariate return distributions and portfolio losses are backtested and compared to the delta-normal model.

The rest of the paper is organized as follows: In section 2 we test statistically whether covariance estimates are time-varying and investigate its sources in volatilities and correlations. The statistical properties of the non-stationary regression model are outlined in the first part of section 3. After dealing with the pitfalls of the nonparametric covariance estimation, we implement a wholistic smoothing conception regarding the 30-dimensional benchmark universe. Section 4 first completes the overall model approximation to the multivariate return series. Second, we provide a forecasting experiment regarding the non-stationary modeling of asset portfolios to assess and evaluate quantiles of future portfolio returns. We conclude in section 5.

<sup>5</sup>The estimation of  $\beta_2$  can be reduced to  $\hat{\beta}_2(\varepsilon.) = \frac{E\varepsilon^4}{E\varepsilon^2}$ . For more details see Gurtler et al. (2009).

## 2 Testing for time-varying covariances

Our data source comprises 30 empirical return series  $\{X_{k,t}\}_{t=1,\dots,n}$  ( $k = 1, \dots, 30$ ) of equity indices, interest rates, credit spreads and exchange rates, corresponding to the data setup of Gürtler and Rauh (2009). The exact description of benchmark indices is provided in table 1, closing prices  $\{P_{k,t}\}_{t=0,\dots,n}$  of common trading days from January 4, 1999 to December 31, 2008 are obtained by Bloomberg data download.<sup>6</sup> In this section we apply a standard return conception where arithmetic returns  $X_t = \frac{P_t - P_{t-1}}{P_{t-1}}$  are used for equities and currencies, and differences of prices  $X_t = P_t - P_{t-1}$  (diff-returns) are used for interest rates and credit spreads (in basis points; bp).<sup>7</sup>

Concerning the time-variation of volatility we refer to Gürtler and Rauh (2012) and the references therein. It is a well established fact, that financial time series are heteroscedastic. The authors provide an empirical survey of the univariate non-stationary regression model, where volatility is estimated non-parametrically for all return series of table 1 by dint of Nadaraya-Watson estimates. In the present paper we examine whether the (unconditional) covariance is time-varying and identify the sources. Thus, we aim at testing, if the correlation  $\text{Corr}(X_k, X_l)$  between time series  $\{X_{k,t}\}_t$  and  $\{X_{l,t}\}_t$  is constant over time. In the case of constant correlations but time-varying covariances, estimates for the latter could be formulated within the non-stationary approach (2) as (hypothesis):

$$\hat{\sigma}_{k,l}^2(t) = \text{Corr}(X_k, X_l) \cdot \hat{\sigma}_k(t) \hat{\sigma}_l(t) \quad (12)$$

where  $\hat{\sigma}_k(t) = \hat{\Sigma}_{k,k}(t)$ .

We first survey the simple empirical estimator for covariance  $\hat{\sigma}_{k,l;T,t}^2 = \frac{1}{T-1} \sum_{i=t-T+1}^t (X_{k,i} - \bar{X}_{k;T,t}) \cdot (X_{l,i} - \bar{X}_{l;T,t})$  of the last  $T$  returns up to time  $t$  (with  $\bar{X}_{k;T,t}$  the empirical mean in  $t$  over window  $T$ ), but with different periodicities  $(i-1, i]$  and lengths  $T$  of the measurement window. Figure 1 depicts the estimated covariances between pairs of four examples, one of each exposure class, from January 2004 to December 2008. The first estimator is based on monthly returns with five years of history, the second estimation is based on the same horizon but with a daily frequency, whereas the third approach is the empirical covariance of daily returns over a one-year moving window (258 days, per average). We observe some differences in the graphs on time segments or the whole horizon, implicating the relevance of data granularity and sample size for covariance and risk measurement. Contrary to a comparatively plain and nearly constant course until 2006, an increase of the linear dynamics is exposed especially in the last two years.

With the beginning of the US Subprime-crisis in 2007, the covariances between major market segments experienced dramatical changes. On the one hand, the covariance between US stocks and the US rates jumped from a zero-level to a strong positive extent regarding the 258-day empirical estimate (a decline of the MSCI North America of more than 50% was accompanied by a decrease of the 5-year US Swap rate from 5.74% to 1.99%). On the other hand, the covariance of the equity index to the exchange rate EUR/USD (price notation) dropped from zero into the negative area (for the benefit of an EUR-investor in US securities). Seemingly dramatical jumps occurred to the covariance between swap spreads (*CredSta*) and the equity-,

<sup>6</sup>In the following we omit the index  $k$  whenever possible for the sake of simplicity.

<sup>7</sup>We refer to Gürtler and Rauh (2009) regarding the impact of different return types. They do not significantly influence our test results.

exposure class	amount	benchmark (BM), description
EquEUR	$w_1$	MSCI Daily TR Gross EMU Local
EquexEUR	$w_2$	MSCI Daily TR Gross Europe ex EMU Local
EquNA	$w_3$	MSCI Daily TR Gross North America Local
EquAsP	$w_4$	MSCI Daily TR Gross (Asia) Pacific Local
EquEM	$w_5$	MSCI Emerging Markets (Free) Local
RateEUR	$w_6$	EUR Swap annual (30/360), 5 year
RateUSD	$w_7$	USD Swap annual (30/360), 5 year
RateJPY	$w_8$	JPY Swap annual (act/365), 5 year
CredSta	$w_9$	synthetic BM, models global spread Govt. to Swap
CredSwa	$w_{10}$	synthetic BM, models Pfandbrief/ Covereds Spread
CredAAA	$w_{11}$	JP Morgan Credit Index AAA Asset Swap Spread
CredAA	$w_{12}$	JP Morgan Credit Index AA Asset Swap Spread
CredA	$w_{13}$	JP Morgan Credit Index A Asset Swap Spread
CredBBB	$w_{14}$	JP Morgan Credit Index BBB Asset Swap Spread
CredEM	$w_{15}$	JP Morgan EMBI Global Divers. Sov. Spread
CredHY	$w_{16}$	Merrill Lynch HY US BB-B (Spread to US-Swap)
CurrEUR	$w_{17}$	synthetic BM, set constant 1 (home currency)
CurrGBP	$w_{18}$	ECB Euro Exchange Ref. Rate as EUR/GBP
CurrCHF	$w_{19}$	ECB Euro Exchange Ref. Rate as EUR/CHF
CurrSEK	$w_{20}$	ECB Euro Exchange Ref. Rate as EUR/SEK
CurrDKK	$w_{21}$	ECB Euro Exchange Ref. Rate as EUR/DKK
CurrNOK	$w_{22}$	ECB Euro Exchange Ref. Rate as EUR/NOK
CurrUSD	$w_{23}$	ECB Euro Exchange Ref. Rate as EUR/USD
CurrCAD	$w_{24}$	ECB Euro Exchange Ref. Rate as EUR/CAD
CurrJPY	$w_{25}$	ECB Euro Exchange Ref. Rate as EUR/JPY
CurrAUD	$w_{26}$	ECB Euro Exchange Ref. Rate as EUR/AUD
CurrNZD	$w_{27}$	ECB Euro Exchange Ref. Rate as EUR/NZD
CurrSGD	$w_{28}$	Bloomberg exchange rate as EUR/SGD
CurrHKD	$w_{29}$	Bloomberg exchange rate as EUR/HKD
CurrEM	$w_{30}$	synthetic BM for exchange rate of EM currencies

Table 1: Exposure classes and corresponding benchmarks.

The exposure conception is exemplified for a European (EUR) investor, the contribution of each asset to the several exposure classes has to be examined. Four general types of exposure classes are distinguished:

- Equity exposures, with subclasses for different economic areas. Exposure amounts  $w_1, \dots, w_5$  are measured as the effective (market valued) investments.
- Interest rate exposures, with subclasses for different currency areas. Exposure amounts  $w_6, \dots, w_8$  are measured as the basis point value (bpv) of attributed securities, with  $w_j$  being negative for long-positions.
- Credit spread exposures, with subclasses for different rating classes or types of coverage. Exposure amounts  $w_9, \dots, w_{16}$  are measured again as the bpv, with  $w_j$  being negative for long-positions.
- Currency exposures, with subclasses for different denominations. Exposure amounts  $w_{17}, \dots, w_{30}$  are measured as the effective investments in home currency.

Liquid market indices are matched to the exposure classes as risk factors. Equity benchmarks are MSCI equity indices (total return) in local currency. Interest rate benchmarks are 5 year swap rates (in bp). Credit spread benchmarks are asset swap spreads (in bp) from JP Morgan, Merrill Lynch or synthetic created (BM *CredSta*: 75% Euro +20% US +5% Japan govt. to swap spread (5y.); BM *CredSwa*: 5y. PEX yield to swap rate). Currency benchmarks are mostly ECB exchange rates in price quotation (BM *CurrEUR*: index constant 1 (to cover non- risky EUR investments), BM *CurrEM*: derived from return differences of MSCI EM index (in EUR) and MSCI EM index (local)). Thanks to Bloomberg and the cited index data providers.



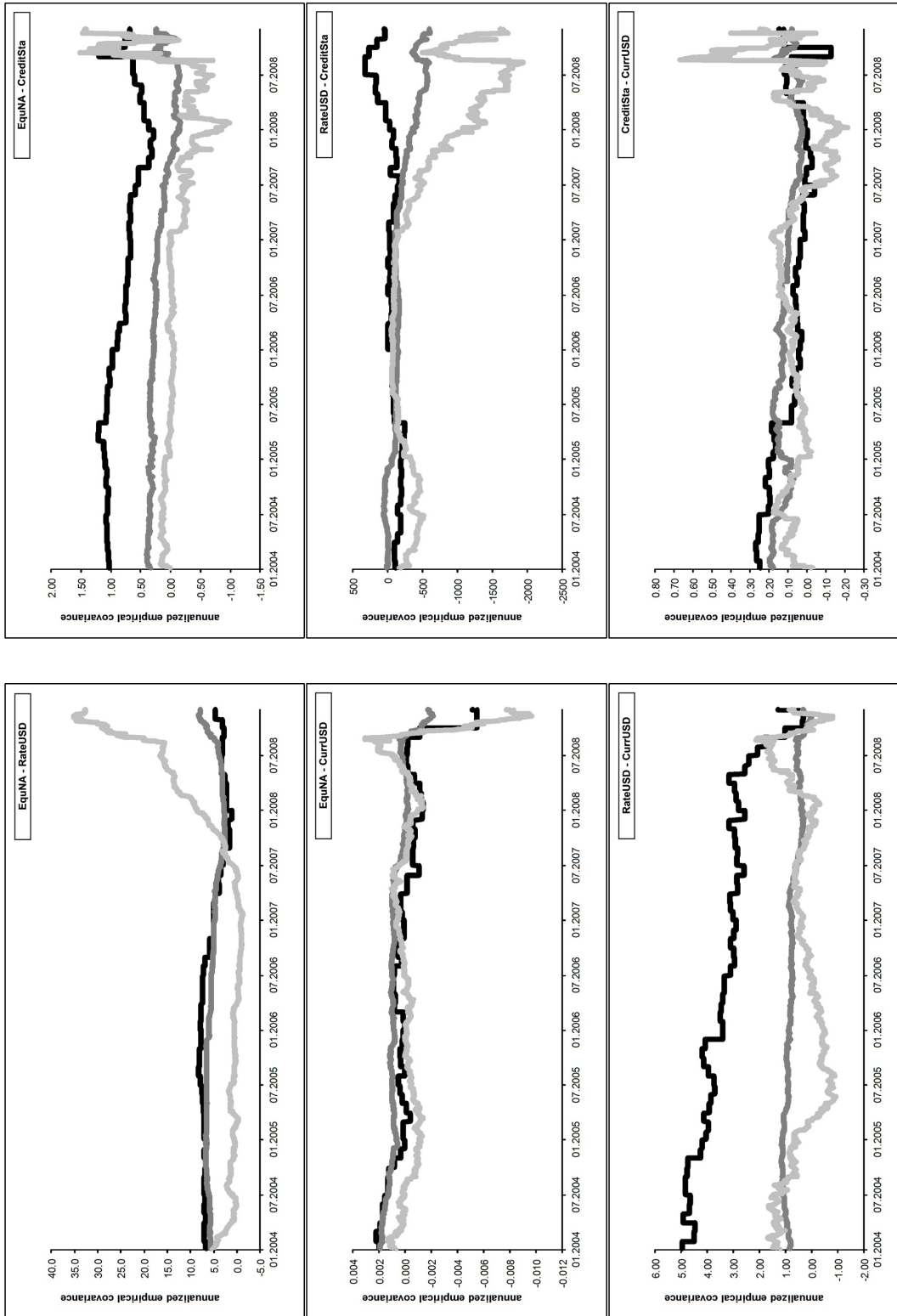


Figure 1: Annualized empirical covariances in 3 setups for periodicity and length of the return series (each picture): 1. monthly returns with five years of history (black line), 2. daily returns over five years (dark gray line), 3. last year's daily returns (bright gray line). The pairs of examples *EquNA*, *RateUSD*, *CreditSta*, *CurrUSD* correspond to table 1.

interest rate- and currency series. Common to all graphs is the smoother development of the estimates with 5 years of history against the one-year moving average. However some steps are observed for the monthly covariances due to the small sample size of 60 returns being vulnerable to singular extreme realizations.<sup>8</sup> As the underlying volatility of singular return series is not (yet) considered, we can not (yet) discriminate in all cases between a shift of the interdependences and the time-varying volatility of assets.

The better interpretation of a time-varying linear dependence is enabled, of course, by the standardized measure of empirical correlations, that are presented in figure 2 for the same examples. Again, the correlation estimators deviate dependent on the length and frequency of returns, and in turn, most dynamics are observed within the years 2007 - 2008. For the *EquNA - RateUSD* example the correlation of 5-year daily returns is nearly constant, whereas one-year daily correlations change their sign from February 2007 ( $-0.17$ ) to October 2008 ( $+0.61$ ). Contrariwise, the long-term monthly correlations decrease in 2007 first (from 0.44 to 0.12) and increase in 2008 again (up to 0.36); this might be founded in losing observations from the last bear market (burst of 'I.T. bubble' in 2000 and economic crisis thereon) at the end of the 5-year window and the reaction time against a few new extreme realizations. For another example, the covariance trend between US stocks and the EUR/USD rate is confirmed by correlations with a sudden decline in the second half of 2008 (especially for 5-year MA: from  $-0.03$  to  $-0.44$ ; 258-day correlation: from  $+0.14$  to  $-0.20$ ). Regarding the *CredSta* pairs some of the effects from the above paragraph recur for empirical correlations (especially for monthly correlations), but are less dramatical and their significance cannot be deduced visually.<sup>9</sup> Concluding, in addition to our indications for heteroscedasticity in terms of covariances and correlations, we require a test criterion whether the time-dependence is statistically significant.

For disjoint subsamples  $\{X_{k,t}, X_{l,t}\}_{t=n_0+1, \dots, n_0+n_1}$  and  $\{X_{k,t}, X_{l,t}\}_{t=n_0+n_1+1, \dots, n_0+n_1+n_2}$ , where  $n_0 \geq 0$ ,  $n_1, n_2 \geq T$  and  $n_0 + n_1 + n_2 \leq n$ , we test whether the corresponding empirical correlations  $\bar{\rho}_{k,l}^{(1)}$  and  $\bar{\rho}_{k,l}^{(2)}$  are time-invariant. Hence, the null hypothesis of the two-sided two-sample test is  $H_0 : \bar{\rho}_{k,l}^{(1)} = \bar{\rho}_{k,l}^{(2)}$ . With a Fisher-transformation  $z^{(i)}$  of correlations and ensuring  $n_1 = n_2$  the test statistic is defined as:

$$Z := \left( z^{(1)} - z^{(2)} \right) \frac{\sqrt{n_1 - 3}}{\sqrt{2}}, \quad z^{(i)} = \frac{1}{2} \ln \left( \frac{1 + \bar{\rho}_{k,l}^{(i)}}{1 - \bar{\rho}_{k,l}^{(i)}} \right). \quad (13)$$

Then the absolute of the normed difference,  $|Z|$ , is approximately normally distributed.  $H_0$  is rejected on the  $\alpha$  level of significance if  $|Z| > z_{1-\frac{\alpha}{2}}$ , where the latter is a standardized Gaussian quantile.

<sup>8</sup>Moreover we deduced a heuristic for an exponential weighted moving average covariance:  $\tilde{\sigma}_{k,l;t}^2 = (1-\lambda)R_{k,t}R_{l,t} + \lambda\tilde{\sigma}_{k,l;t-1}^2$  (with centered returns  $R_{.,t}$  and decay factor  $\lambda \in (0, 1)$ ), that is a consistent estimate for covariance (asymptotically unbiased for an infinite (granulated) series, its variance converges to zero ( $\lambda \nearrow 1$ )). With  $\lambda = 0.94$ , as recommended by RiskMetrics<sup>TM</sup> for daily returns, very undersmoothed estimates result with a sawtooth-structure at shocks. We do not display them in figure 1 since a much wider scale was required, with the consequence that dynamics of the other estimates appear negligible. However the adequate weighting of recent/past returns is a further task in covariance estimation, that is answered in the nonparametric conception later.

<sup>9</sup>Again we implemented a decay estimate of correlations:  $\check{\rho}_{k,l;t} = \frac{(1-\lambda)R_{k,t}R_{l,t} + \lambda\tilde{\sigma}_{k,l;t-1}^2}{\sqrt{(1-\lambda)R_{k,t}^2 + \lambda\tilde{\sigma}_{k,t-1}^2} \cdot \sqrt{(1-\lambda)R_{l,t}^2 + \lambda\tilde{\sigma}_{l,t-1}^2}}$ . Once more, exponential smoothing (with  $\lambda = 0.94$ ) presents a strong oscillating graph that uses in singular points almost the whole range  $[-1, 1]$  of correlations. Instead of optimizing the decay-approach we provide with the kernel regression within the non-stationary approach an enhanced method. Already at this point we can conclude a trade-off between a longer history of returns and the weighting of recent returns.

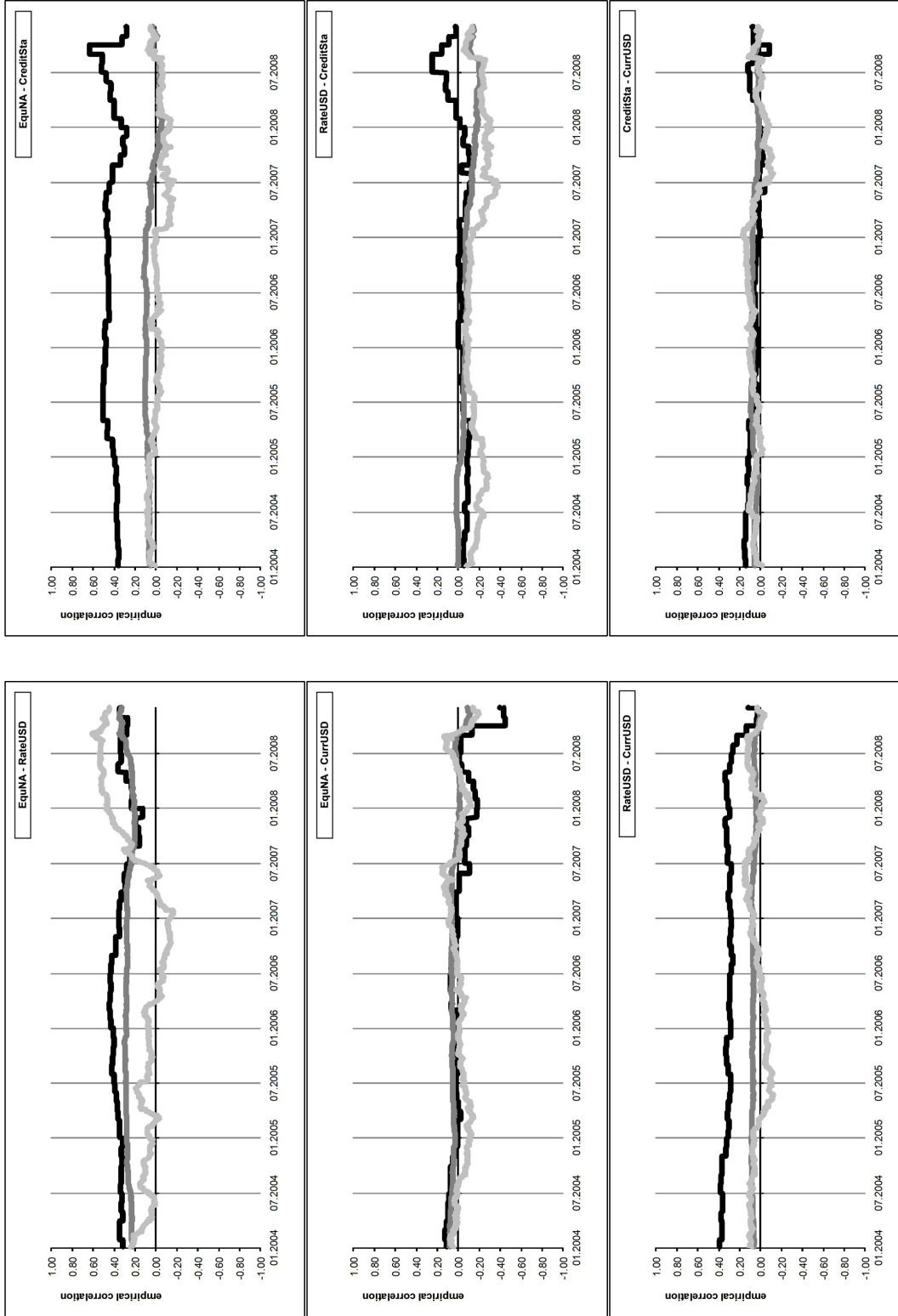


Figure 2: Empirical correlations in 3 setups for periodicity and length of the return series (each picture): 1. monthly returns with five years of history (black line), 2. daily returns over five years (dark gray line), 3. last year's daily returns (bright gray line). The pairs of examples *EquiNA*, *RateUSD*, *CreditSta*, *CurrUSD* correspond to table 1.

We implement the test for subsamples regarding the 5-year monthly-, the 5-year daily- and the 1-year daily empirical correlations of each benchmark combination from table 1 (except *CurrEUR*, because it is used as deterministic numeraire; hence having 406 correlation pairs). In each case we test whether empirical correlations at January 2, 2004  $\bar{\rho}_{k,l}^{(01/2004)}$  and December 31, 2008  $\bar{\rho}_{k,l}^{(12/2008)}$  are identical. Regarding the short-term daily estimates we additionally test for a mid point July 3, 2006 whether  $\bar{\rho}_{k,l}^{(01/2004)} = \bar{\rho}_{k,l}^{(07/2006)}$  and  $\bar{\rho}_{k,l}^{(07/2006)} = \bar{\rho}_{k,l}^{(12/2008)}$ . Moreover, we check for all setups if the maximum correlation of the whole sample  $\bar{\rho}_{k,l}^{(\max)}$  differs significantly from the minimum  $\bar{\rho}_{k,l}^{(\min)}$  using the above test criterion; although this min-max method compares correlations of not necessarily disjoint return sets, its rejection might be even more an indication for time-dependency. As a summary heuristic we developed the criterion:

$$H_0 \text{ is overall rejected if at least 1 test is not passed,}$$

which is quite conservative as it is directly linked to the success of the min-max criterion.<sup>10</sup> The test results for the pairs of the four above, representative benchmarks are provided in figure 3. The summary test heuristics of all benchmark combinations are consolidated in figure 4.

Fisher's z-test on time-invariant empirical correlations						
	EquNA-RateUSD	EquNA-CreditSta	EquNA-CurrUSD	RateUSD-CreditSta	RateUSD-CurrUSD	CreditSta-CurrUSD
<b>5-year monthly</b>						
c-value subsequent	1.96	1.96	1.96	1.96	1.96	1.96
a. 01/2004 vs. 12/2008	0.1061	0.4411	2.8177	0.4365	1.5505	0.3434
b. min vs. max	1.8682	2.4878	3.2221	2.0405	2.0166	1.2996
-> rejection	no	yes	yes	yes	yes	no
<b>5-year daily</b>						
c-value subsequent	1.96	1.96	1.96	1.96	1.96	1.96
a. 01/2004 vs. 12/2008	2.7352	0.2901	4.2087	3.8602	0.8209	1.0566
b. min vs. max	4.3834	4.6637	4.9871	5.9722	2.2414	2.3214
-> rejection	yes	yes	yes	yes	yes	yes
<b>1-year daily</b>						
c-value subsequent	1.96	1.96	1.96	1.96	1.96	1.96
a. 01/2004 vs. 12/2008	1.9633	0.6043	1.7968	0.4389	0.2615	0.4039
b. 01/2004 vs. 07/2006	4.0264	0.2858	0.5371	0.1099	0.5978	1.2698
c. 07/2006 vs. 12/2008	5.9897	0.8901	1.2598	0.3290	0.3364	0.8659
d. min vs. max	10.0038	3.0645	4.0853	3.9584	3.1710	3.3356
-> rejection	yes	yes	yes	yes	yes	yes

Figure 3: Tests for time-invariance of all correlation pairs of *EquNA*, *RateUSD*, *CredSta*, *CurrUSD* regarding the three setups for periodicity and length of the return series. Test rejections are gray highlighted.

number of rejected tests for time-invariant correlations (real correlation pairs: 406)	
empirical estimates based on	rejections
-> 5-year monthly returns	255
-> 5-year daily returns	382
-> 1-year daily returns	406

Figure 4: Summary of test results for time-invariant correlations regarding all pairs of benchmark return series from table 1 and the three setups for periodicity and length.

The test results confirm the visual appraisals of the examples in figure 2: While the invariance of correlations is neglected for about 63% of the monthly return series, the refinement to daily returns leads to

<sup>10</sup>On the one hand, the different time-setups are best comparable with this heuristic. On the other hand, an erratic shape might be missed by concentrating only on the start (mid) and end point. For some series we observed that, coincidentally, correlations in these two (three) points are quite similar, while the span was much larger in between.

rejections for almost all (94%) of the long-term daily measurements. Constant correlations are rejected for all the one-year moving-averages. Regarding the pairs of our four examples only monthly correlations of *EquNA* to *RateUSD* and *CredSta* to *CurrUSD* are accepted as time-invariant. Most of the long-term daily correlations have strong support to be originated from heteroscedastic multivariate processes (since test statistics are much greater than the critical values). Moreover, strong hypothesis rejections result on the 1-year dynamics, but that follows only from the min-max criterion for all series except *EquNA - RateUSD*.<sup>11</sup> Altogether we conclude that (unconditional) correlations or covariances change significantly over time, especially for daily returns. The sample size and periodicity of the data base (and the weighting of recent realizations) have an important impact on the estimates and test results. Furthermore, hypothesis (12) has to be neglected and a direct estimation of (nonparametric) covariances  $\hat{\sigma}_{k,l}^2(t)$  will be necessary. A time-dependent, non-stationary risk interdependence should be modeled for financial time series.

### 3 Nonparametric covariance estimation in the non-stationary model

In the following we concentrate on the daily benchmark series since the larger statistical evidence for time-varying correlations was brought on the daily data base. Moreover, we work below with the preferred return conception of Grtler and Rauh (2009), where log-returns  $X_t = \ln P_t - \ln P_{t-1}$  are used for equities and currencies; the return type diff-returns remains unchanged for interest rates and credit spreads. In this section we first introduce two ways of smoothing concerning the regression in the non-stationary model (2) and outline the necessary statistical results. Second, we turn to the practical application of nonparametric covariance estimation in terms of both smoothing conceptions and the benchmark universe from table 1.

#### 3.1 Wholistic vs. individual smoothing: theoretical preparation

Before outlining the statistical properties of the nonparametric estimation  $\hat{\Sigma}^2(t)$  or  $\hat{\Sigma}_{(1)}^2(t)$  of covariance matrix  $\mathbf{S}_t \mathbf{S}'_t = \Sigma^2(t) \in \mathbb{R}^{d \times d}$  we emphasize two general alternatives of smoothing:

- (a) Wholistic smoothing of all elements of the covariance matrix using one bandwidth  $h \in \mathbb{N}$ , according to presentations (5) and (6) for the MNWE,
- (b) Individual smoothing of the covariances between  $R_{k,t}$  and  $R_{l,t}$  (for all  $k, l = 1, \dots, d$ ) with bandwidths  $h_{k,l} \in \mathbb{N}$  from matrix  $\mathbf{h} = (h_{k,l})_{i,j} \in \mathbb{R}^{d \times d}$ . The corresponding pairwise NWE are:

(I) Two-sided PNWE (smoother):

$$\hat{\sigma}_{k,l}^2(t) := \frac{\sum_{i=1}^n K_{h_{k,l}}(i-t) R_{k,i} R_{l,i}}{\sum_{i=1}^n K_{h_{k,l}}(i-t)}, \quad (14)$$

(II) One-sided PNWE (filter):

$$\hat{\sigma}_{(1)k,l}^2(t) := \frac{\sum_{i=1}^t K_{h_{k,l}}(i-t) \tilde{R}_{k,i} \tilde{R}_{l,i}}{\sum_{i=1}^t K_{h_{k,l}}(i-t)}, \quad (15)$$

<sup>11</sup>As established above, the min-max method is an effective criterion, whereas the single, disjoint points do not reflect the whole span of heteroscedasticity. Nevertheless testing empirical correlations solely at start (mid) and end points results in 104 (regarding 5-year monthly returns), 235 (5-year daily) and 275 (1-year daily) rejections.

and  $\hat{\Sigma}_{\mathbf{h}}^2(t)$  or  $\hat{\Sigma}_{(1)\mathbf{h}}^2(t)$  label the respective matrix of estimates.

For proving consistency of the covariance estimators an asymptotic framework is required, that enables an increase of the frequency for observing data points on a fixed time-frame and scans the (unknown) regression function more and more precisely. A transformed multiplicative return model on a unit design  $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1$  with adopted function values  $\mathbf{S}\left(\frac{t}{n}\right) = \mathbf{S}(t)$  is used by Gurtler et al. (2009):

$$\begin{aligned} \mathbf{X}_{t,n} &= \boldsymbol{\mu} + \mathbf{S}\left(\frac{t}{n}\right) \boldsymbol{\varepsilon}_{t,n}, \quad t = 1, \dots, n, \\ \boldsymbol{\varepsilon}_{1,n}, \dots, \boldsymbol{\varepsilon}_{n,n} &\text{ iid rvecs with independent coordinates, } E\boldsymbol{\varepsilon}_{k,1,n} = 0, \text{ Var}\boldsymbol{\varepsilon}_{k,1,n} = 1 \ (\forall k = 1, \dots, d), \\ \mathbf{S}\left(\frac{t}{n}\right) &\text{ is an invertible } \mathbb{R}^{d \times d} \text{ matrix and smooth function of time.} \end{aligned} \quad (16)$$

The two-sided transformed MNWE  $\hat{\Sigma}_{h_n}^2(u)$  and the one-sided MNWE  $\hat{\Sigma}_{(1)h_n}^2(u)$  for estimating the covariance matrix  $\mathbf{S}(u)\mathbf{S}'(u) = \Sigma^2(u)$ ,  $u \in [0, 1]$ , are structurally analogous to equations (5) and (6). Under certain regularity conditions<sup>12</sup> on the kernel  $K$ , the bandwidth  $h_n$  and the smoothness of the covariance function  $\Sigma^2(\cdot)$ , the following propositions regarding the MNWE of the wholistic approach hold:

- (P1) Under the regularity conditions and setup (16) the sequence  $\left(\hat{\Sigma}_{h_n}^2(u)\right)_{n \in \mathbb{N}}$  of estimators for  $\Sigma^2(u)$  is consistent for all  $u \in (0, 1)$ .
- (P2) Under the regularity conditions and setup (16) the sequence  $\left(\hat{\Sigma}_{(1)h_n}^2(u)\right)_{n \in \mathbb{N}}$  of estimators for  $\Sigma^2(u)$  is consistent for all  $u \in (0, 1]$ .

Fully analytical proofs are provided in Gurtler et al. (2009). From (P1) follows that the sequence of estimates is asymptotically unbiased at interior points of  $[0, 1]$  and its variance converges to zero for an infinitesimal design ( $n \rightarrow \infty$ ). The authors do not derive an asymptotic normal distribution for the sequence of estimators  $\left(\hat{\Sigma}_{h_n}^2(u)\right)_{n \in \mathbb{N}}$ , but they conclude the existence and a unique solution from convergence results of  $E\hat{\Sigma}_{h_n}^2(u)$  and  $\text{Var}\hat{\Sigma}_{h_n}^2(u)$ .<sup>13</sup>

Since  $\hat{\Sigma}_{h_n}^2(u)$  is unbiased in its components with a limiting variance of zero, consistency remains valid for NWEs of covariance pairs. Hence,  $\hat{\sigma}_{h_n;k,l}^2(u)$  (individual smoother in transformed version) is a consistent estimate for  $\Sigma_{k,l}^2(u)$  for all  $u \in (0, 1)$ , provided that bandwidths  $h_{n;k,l}$  fulfill the regularity conditions. Nevertheless the consistency of the matrix estimation  $\hat{\Sigma}_{\mathbf{h}_n}^2(u)$  with bandwidth matrix  $\mathbf{h}_n \in \mathbb{R}^{d \times d}$  for individual smoothing does not automatically follow. The basic requirement of a covariance matrix to be

<sup>12</sup>(C1) Let  $K : \mathbb{R} \rightarrow [0, \infty)$  be a symmetrical density with compact support  $[-1, 1]$ , i.e. (i)  $K(v) = 0 \ \forall v \notin [-1, 1]$ , (ii)  $\int_{-\infty}^{\infty} K(v)dv = 1$ , (iii)  $\int_{-\infty}^{\infty} vK(v)dv = 0$ ; (C2) Let  $K$  be continuous with a limited first derivation  $K'$ ; (C3)  $K_{h_n}(\cdot) = \frac{1}{h_n}K\left(\frac{\cdot}{h_n}\right)$  with restrictions to the bandwidth: (i)  $h_n \xrightarrow{n \rightarrow \infty} 0$ , (ii)  $nh_n, \dots, nh_n^4 \xrightarrow{n \rightarrow \infty} \infty, nh_n^6, nh_n^7, \dots \xrightarrow{n \rightarrow \infty} 0$ , (iii)  $nh_n^5 \xrightarrow{n \rightarrow \infty} C^2 \geq 0$ ; (C4') Let the matrix  $\Sigma^2$  be two times continuous differentiable in its elements; (C5') Let random vectors  $\boldsymbol{\varepsilon}_{1,n}, \dots, \boldsymbol{\varepsilon}_{n,n}$  be iid with independent coordinates,  $E\boldsymbol{\varepsilon}_{k,1,n} = 0$ ,  $\text{Var}\boldsymbol{\varepsilon}_{k,1,n} = 1$  and  $E|\boldsymbol{\varepsilon}_{k,1,n}|^{4+\delta_k} < \infty$  for a  $\delta_k > 0 \ \forall k = 1, \dots, d$  and  $n \in \mathbb{N}$ . Concerning the one-sided MNWE condition (C3) has to be replaced with: (C3')  $K_{h_n}(\cdot) = \frac{1}{h_n}K\left(\frac{\cdot}{h_n}\right)$  with restrictions: (i)  $h_n \xrightarrow{n \rightarrow \infty} 0$ , (ii)  $nh_n, nh_n^2 \xrightarrow{n \rightarrow \infty} \infty, nh_n^4, nh_n^5, \dots \xrightarrow{n \rightarrow \infty} 0$ , (iii)  $nh_n^3 \xrightarrow{n \rightarrow \infty} D^2 \geq 0$ .

<sup>13</sup>The sequence  $\sqrt{nh_n}\left(\hat{\Sigma}_{h_n}^2(u) - \Sigma^2(u)\right) \xrightarrow{D} \mathcal{N}^{d \times d}(\boldsymbol{\beta}(u), \boldsymbol{\tau}^2(u)) \ \forall u \in (0, 1)$  with  $\boldsymbol{\beta}(u) = \frac{C}{2}(\Sigma^2(u))'' \int_{-1}^1 v^2 K(v)dv \in \mathbb{R}^{d \times d}$ ,  $C^2 = \lim_{n \rightarrow \infty} nh_n^5 \in \mathbb{R}_0^+$  and  $\boldsymbol{\tau}^2(u) = \mathbf{V} \odot \Sigma^4(u) \int_{-1}^1 K^2(v)dv \in \mathbb{R}^{d \times d}$ ,  $\mathbf{V} \in \mathbb{R}^{d \times d}$  appropriate, element-wise multiplied. The notation  $\mathcal{N}^{d \times d}$  means each element of the  $d \times d$  matrix to be Gaussian distributed. In the case  $d = 1$  the asymptotic normality is obtained with  $V := E\varepsilon_{1,n}^4 - 1$ ; compare Gurtler et al. (2009).

positive semidefinite might be violated due to different bandwidths. Concerning the MNWE it holds with  $d$ -dimensional column vectors  $\mathbf{z}$  that

$$\begin{aligned} \mathbf{z}' \hat{\Sigma}_{h_n}^2(u) \mathbf{z} &= \mathbf{z}' \left( \sum_{i=1}^n w_{h_n;i,n}(u) \mathbf{R}_{i,n} \mathbf{R}'_{i,n} \right) \mathbf{z}, \text{ where } w_{h_n;i,n}(u) := \frac{K_{h_n}\left(\frac{i}{n} - u\right)}{\sum_{i=1}^n K_{h_n}\left(\frac{i}{n} - u\right)} \\ &= \sum_{i=1}^n w_{h_n;i,n}(u) \mathbf{z}' \mathbf{R}_{i,n} \mathbf{R}'_{i,n} \mathbf{z} = \sum_{i=1}^n w_{h_n;i,n}(u) (\mathbf{z}' \mathbf{R}_{i,n})^2 \geq 0. \end{aligned} \quad (17)$$

Factoring out  $w_{h_n;i,n}(u)$  is only possible if it is constant for all matrix elements, or equivalently if  $h_n$  is wholistic. Otherwise  $\mathbf{z}' \hat{\Sigma}_{h_n}^2(u) \mathbf{z} \geq 0$  does not hold in general for individual bandwidths  $h_n$ . The consistency of a portfolio variance  $\boldsymbol{\omega}' \hat{\Sigma}_{h_n}^2(u) \boldsymbol{\omega}$ , where  $\boldsymbol{\omega} \in \mathbb{R}^d$ ,  $\omega_i \geq 0$  and  $\sum_{i=1}^d \omega_i = 1$ , follows only for wholistic smoothing. In contrast, at individual smoothing the positive semidefiniteness and the standardization to correlations  $\hat{\rho}_{h_n;k,l}(u)$  (in a way according to Pearson correlations) are endangered and remain methodically flawed since no functional between  $h_{n;k,l}$ ,  $h_{n;k,k}$  and  $h_{n;l,l}$  is described.

The consistency result of the two-sided estimate can be transmitted to the one-sided counterpart  $\hat{\Sigma}_{(1)h_n}^2(u)$ , conditioned on a faster convergence rate of the wholistic bandwidth  $h_n$ . Additionally (P2) is not only valid at interior points of  $[0, 1]$ , but also at the right frontier. This makes the estimator consistently applicable for forecasting covariance and multivariate return distributions.<sup>14</sup> Again it can be concluded on the asymptotic normal distribution for the sequence of estimators  $\left( \hat{\Sigma}_{(1)h_n}^2(u) \right)_{n \in \mathbb{N}}$ , having a unique solution.<sup>15</sup>

In turn, consistency is valid for NWEs of covariance pairs and some results can be transferred to individual smoothing of matrix elements. The transformed PNWE  $\hat{\sigma}_{(1)h_n;k,l}^2(u)$  is a consistent estimate for  $\Sigma_{k,l}^2(u)$  for all  $u \in (0, 1]$ , subject to  $h_{n;k,l}$  sustaining the bandwidth conditions. However the matrix estimation  $\hat{\Sigma}_{(1)h_n}^2(u)$  at individual smoothing is not necessarily consistent, because the positive semidefiniteness of the covariance matrix might be violated. The analogon to equation (17) holds in general only for a wholistic one-sided bandwidth  $h_n$ . The consistency of a portfolio variance  $\boldsymbol{\omega}' \hat{\Sigma}_{(1)h_n}^2(u) \boldsymbol{\omega}$  and a well-defined correlation matrix immediately follow for wholistic smoothing, but not generally for the individual conception.

For the purpose of kernel regression a polynomial of fourth degree (also called biweight kernel) is established,

$$K(u) := \begin{cases} \frac{15}{16}(1 - u^2)^2 & , \quad |u| \leq 1 \\ 0 & , \quad \text{else} \end{cases}, \quad (18)$$

as an adequate kernel, that satisfies the regularity conditions and is applied in the sequel. The task of bandwidth selection is solved with cross-validation (CV) for the two-sided- and the one-sided setup. In terms of the individual smoothing conception, CV-functions regarding volatility estimation over a discrete design  $\{1, \dots, n\}$  from Gürtler et al. (2009) can be directly carried forward. Optimal bandwidths  $h_{k,l}^{CV}$  for pairwise covariance estimates (14) and (15) are detected with the help of:

<sup>14</sup>On the other hand, within historical samples a bigger estimation error results from the one-sided estimate due to its information lack compared to the two-sided counterpart.

<sup>15</sup>The sequence  $\sqrt{nh_n} \left( \hat{\Sigma}_{(1)h_n}^2(u) - \Sigma^2(u) \right) \xrightarrow{D} \mathcal{N}^{d \times d} \left( \boldsymbol{\beta}_{(1)}(u), \boldsymbol{\tau}_{(1)}^2(u) \right) \forall u \in (0, 1]$  with  $\boldsymbol{\beta}_{(1)}(u) = 2D \left( \Sigma^2(u) \right)' \int_{-1}^0 v K(v) dv \in \mathbb{R}^{d \times d}$ ,  $D^2 = \lim_{n \rightarrow \infty} nh_n^3 \in \mathbb{R}_0^+$  and  $\boldsymbol{\tau}_{(1)}^2(u) = 4\mathbf{V} \odot \Sigma^4(u) \int_{-1}^0 K^2(v) dv \in \mathbb{R}^{d \times d}$ ,  $\mathbf{V} \in \mathbb{R}^{d \times d}$  appropriate, element-wise multiplied. For  $d = 1$  the asymptotic result is obtained with  $V := E\varepsilon_{1,n}^4 - 1$ ; see Gürtler et al. (2009).

(I) Two-sided setup:

$$\begin{aligned}\widetilde{\text{CV}}(h_{k,l}) &= \frac{1}{n} \sum_{j=1}^n \left( R_{k,j} R_{l,j} - \hat{\sigma}_{k,l}^{2(j)}(j) \right)^2, \text{ with } \hat{\sigma}_{k,l}^{2(j)}(j) = \frac{\sum_{i=1, i \neq j}^n K_{h_{k,l}}(i-j) R_{k,i} R_{l,i}}{\sum_{i=1, i \neq j}^n K_{h_{k,l}}(i-j)} \\ &= \frac{1}{n} \sum_{j=1}^n \left( \frac{\sum_{i=1}^n K_{h_{k,l}}(i-j) (R_{k,j} R_{l,j} - R_{k,i} R_{l,i})}{\sum_{i=1, i \neq j}^n K_{h_{k,l}}(i-j)} \right)^2 \xrightarrow{!} \min_{h_{k,l} > 1}\end{aligned}\quad (19)$$

(II) One-sided setup:

$$\begin{aligned}\widetilde{\text{CV}}_{(1)}(h_{k,l}) &= \frac{1}{n-1} \sum_{j=2}^n \left( \tilde{R}_{k,j} \tilde{R}_{l,j} - \hat{\sigma}_{(1)k,l}^{2(j)}(j) \right)^2, \text{ with } \hat{\sigma}_{(1)k,l}^{2(j)}(j) = \frac{\sum_{i=1}^{j-1} K_{h_{k,l}}(i-j) \tilde{R}_{k,i} \tilde{R}_{l,i}}{\sum_{i=1}^{j-1} K_{h_{k,l}}(i-j)} \\ &= \frac{1}{n-1} \sum_{j=2}^n \left( \frac{\sum_{i=1}^{j-1} K_{h_{k,l}}(i-j) (\tilde{R}_{k,j} \tilde{R}_{l,j} - \tilde{R}_{k,i} \tilde{R}_{l,i})}{\sum_{i=1}^{j-1} K_{h_{k,l}}(i-j)} \right)^2 \xrightarrow{!} \min_{h_{k,l} > 1}\end{aligned}\quad (20)$$

The CV-optimal bandwidths  $h_{k,l}^{\text{CV}}$  and  $h_{k,l}^{\text{CV}(1)}$  for the two- and one-sided PNWEs are numerically found via a value table or via analyzing the CV-plot. The matrix of optimal bandwidths  $\mathbf{h}^{\text{CV}} = \left( h_{k,l}^{\text{CV}} \right)_{i,j}$  follows in each setup.

For deriving a global optimal bandwidth  $h \in \mathbb{N}$  in the wholistic smoothing conception Grtler et al. (2009) introduce a metric  $d(\mathbf{A}, \mathbf{B}) := \sum_{k=1}^d \sum_{l=1}^k (A_{k,l} - B_{k,l})^2$  for the distance of symmetric matrices  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d \times d}$ . In terms of the CV-functions we result in:

(I) Two-sided setup:

$$\begin{aligned}\text{CV}(h) &= \frac{1}{n} \sum_{j=1}^n \left[ \sum_{k=1}^d \sum_{l=1}^k \left( R_{k,j} R_{l,j} - \hat{\Sigma}_{k,l}^{2(j)}(j) \right)^2 \right], \text{ as } \hat{\Sigma}_{k,l}^{2(j)}(j) = \frac{\sum_{i=1, i \neq j}^n K_h(i-j) R_{k,i} R_{l,i}}{\sum_{i=1, i \neq j}^n K_h(i-j)} \\ &= \frac{1}{n} \sum_{j=1}^n \left[ \sum_{k=1}^d \sum_{l=1}^k \left( \frac{\sum_{i=1}^n K_h(i-j) (R_{k,j} R_{l,j} - R_{k,i} R_{l,i})}{\sum_{i=1, i \neq j}^n K_h(i-j)} \right)^2 \right] \\ &= \sum_{k=1}^d \sum_{l=1}^k \widetilde{\text{CV}}_{k,l}(h) \xrightarrow{!} \min_{h > 1}\end{aligned}\quad (21)$$

(II) One-sided setup:

$$\begin{aligned}\text{CV}_{(1)}(h) &= \frac{1}{n-1} \sum_{j=2}^n \left[ \sum_{k=1}^d \sum_{l=1}^k \left( \tilde{R}_{k,j} \tilde{R}_{l,j} - \hat{\Sigma}_{(1)k,l}^{2(j)}(j) \right)^2 \right], \hat{\Sigma}_{(1)k,l}^{2(j)}(j) = \frac{\sum_{i=1}^{j-1} K_h(i-j) \tilde{R}_{k,i} \tilde{R}_{l,i}}{\sum_{i=1}^{j-1} K_h(i-j)} \\ &= \frac{1}{n-1} \sum_{j=2}^n \left[ \sum_{k=1}^d \sum_{l=1}^k \left( \frac{\sum_{i=1}^{j-1} K_h(i-j) (\tilde{R}_{k,j} \tilde{R}_{l,j} - \tilde{R}_{k,i} \tilde{R}_{l,i})}{\sum_{i=1}^{j-1} K_h(i-j)} \right)^2 \right] \\ &= \sum_{k=1}^d \sum_{l=1}^k \widetilde{\text{CV}}_{(1)k,l}(h) \xrightarrow{!} \min_{h > 1}\end{aligned}\quad (22)$$



The last equality follows each with the permutation of sums and expressions for  $\widetilde{CV}_{k,l}(h)$  and  $\widetilde{CV}_{(1)k,l}(h)$  that resemble the pairwise CV. The wholistic CV-measure requires homogeneous return types and asset categories.<sup>16</sup> Of course, it is suboptimal to determine the respective (individual) optimal bandwidths  $h^{CV,k,l} = h_{k,l}^{CV}$  first and to work with their average

$$\bar{h}_{(\cdot)}^{CV} := \frac{2}{d^2 + d} \sum_{k=1}^d \sum_{l=1}^k h_{k,l}^{CV}. \quad (23)$$

But it generates a simplified heuristic for the sake of runtime and memory capacity for large, heterogeneous empirical samples. Later on we achieve a transition from individual to wholistic smoothing with a similar heuristic.

### 3.2 Wholistic vs. individual smoothing: practical application

In the following we approximate the non-stationary regression model (2) to the complete data base regarding table 1 and describe the interrelated dynamics of the 30 daily benchmark series from January 1999 to December 2008 (2576 return observations). A special focus is set on the nonparametric estimation of covariances. For the beginning we apply the individual smoothing conception for covariance pairs  $\sigma_{k,l}^2(t)$  and analyze the development of PNWEs (14) and (15) over time. Figure 5 presents the nonparametric covariance estimators for the same examples (pairs of *EquNA*, *RateUSD*, *CredSta* and *CurrUSD*) as empirically analyzed and tested in section 2.

Before describing the single graphs, let us share some considerations to bandwidth selection. In general we work with the (pairwise) cross-validation method as introduced in the previous section (formulas (19) and (20)) to obtain optimal bandwidths  $h_{k,l}^{CV}$  and  $h_{k,l}^{CV(1)}$ . But this leads only in about 54% of the cases to automatically suitable bandwidths (regarding all results of the two- and one-sided implementation for daily returns from 1999 to 2008), whereas the others are extremes at the boundaries of our bandwidths grid  $\{2, \dots, 200\}$ . We restrict the grid to a maximum due to sometimes plane or slowly declining CV-graphs; furthermore, we try to avoid oversmoothing and need a bandwidth lower one year for comparisons with smaller samples (and limiting the boundary affected region).<sup>17</sup> For bandwidth adjustments we considered two possibilities: 1. Analysing the CV-graph for local minima or a range of minima; 2. Using the maximum of bandwidths from corresponding volatility estimates,  $h_{k,l}^{opt} = \max \{h_k^{opt}, h_l^{opt}\}$ .<sup>18</sup> For reasons of automatization in estimating a  $30 \times 30$  covariance matrix (435 covariance pairs, 30 variances) we execute the second alternative for bandwidth adjustments (if necessary) and denote  $h_{k,l}^{opt}$  and  $h_{(1)k,l}^{opt}$  as the final optimal bandwidth.

<sup>16</sup>Counterexample: Modeling the bivariate dynamics of an equity (using log-returns of prices, in %) and an interest rate (using differences of rates, in basispoints) will generate different dimensions of individual CV-functions. As those are inserted in  $CV \cdot (h)$ , its minimization might be dominated by a single component and is disputable at the end.

<sup>17</sup>Moreover, we require one-sided bandwidths to be  $h_{k,l}^{CV(1)} \geq 6$  to ensure an adequate degree of smoothness; it was automatically fulfilled in this setup for all  $h_{k,l}^{CV(1)} \neq 2$ . At the repeats of the analyses for daily returns setups 1999-2006, 1999-2000, 2005-2006 and for monthly return setups 1999-2006, 1999-2008 we obtain an average CV success rate of about 52%.

<sup>18</sup>Alternative 2 requires optimal two-sided or one-sided bandwidths  $h_k^{opt}, h_l^{opt}$  of volatility estimators. According to Gürtler and Rauh (2009) we compensate a failed CV with respect to other horizons, similar asset series or the one-/ two-sided equivalent. This method is easy to implement for large data sets, but alternative 1 is more systematic and to be preferred in general.

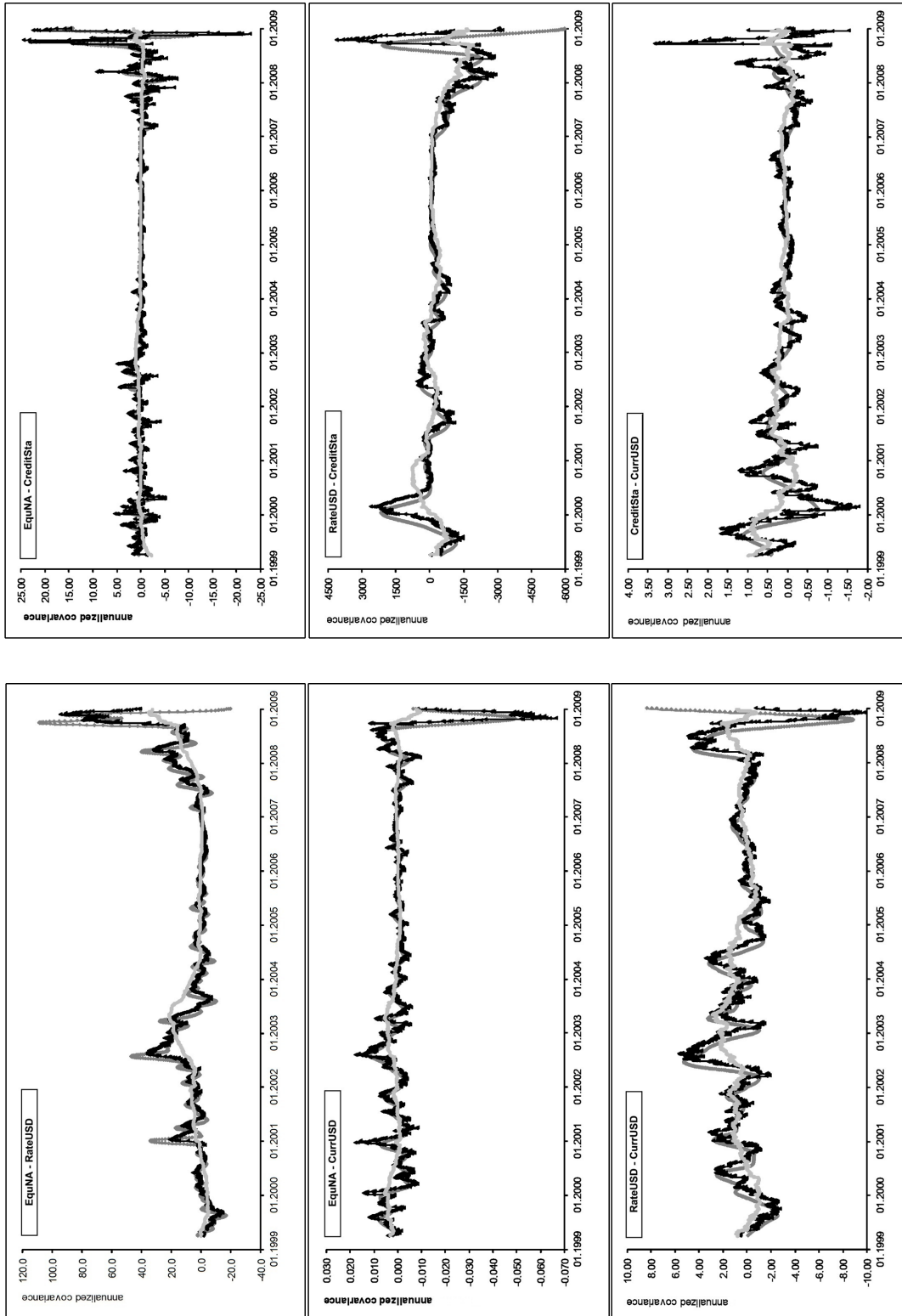


Figure 5: Annualized covariance estimates regarding daily returns of the pairs of series *EquiNA*, *RateUSD*, *CreditSta* and *CurrUSD*. From the individual smoothing of regression model (2) the two-sided PNWEs are depicted in dark gray bullets and the one-sided PNWEs in black triangles. The 258-day empirical covariance (bright gray line) is opposed each with. For used bandwidths see figure 6.

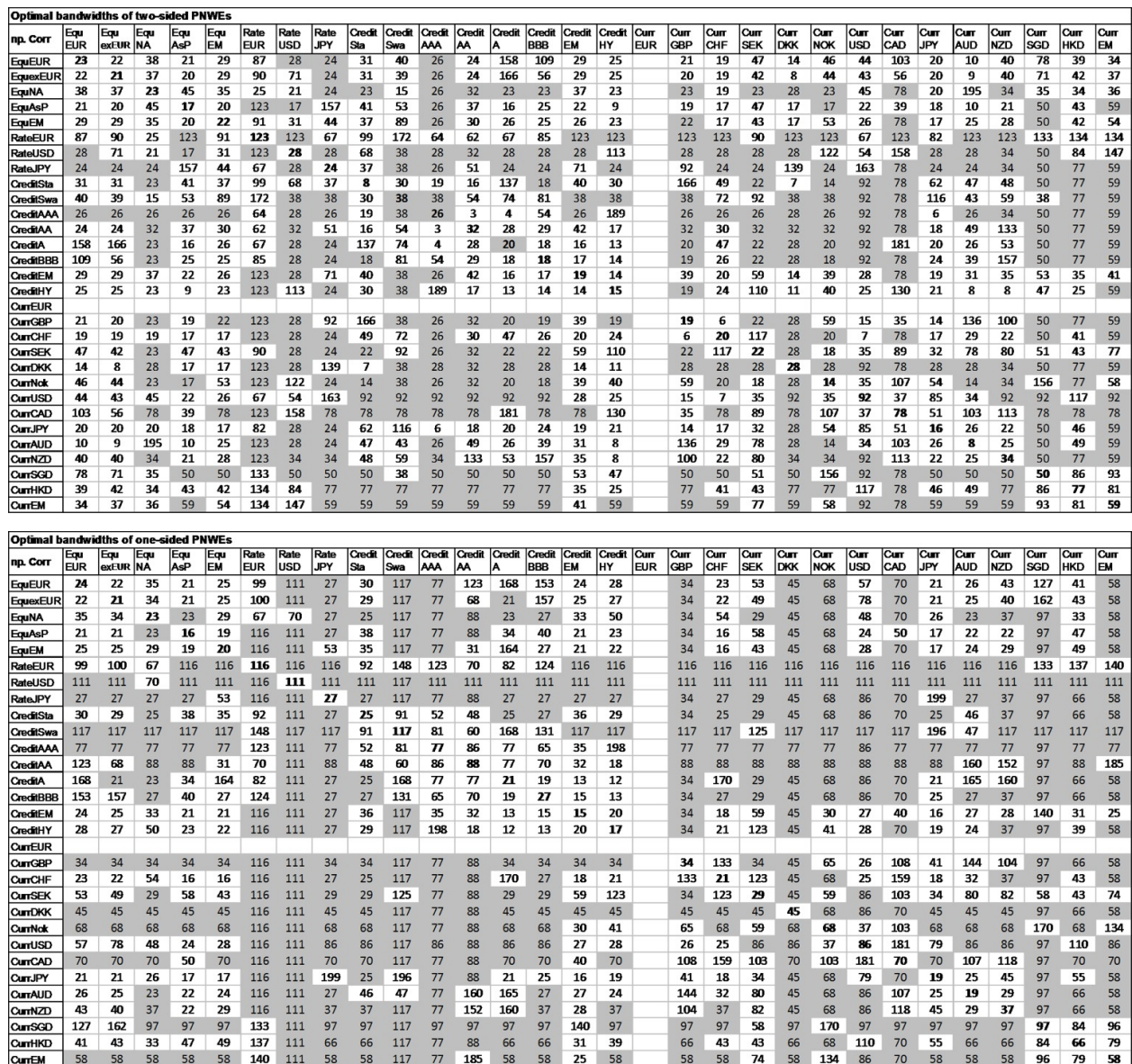


Figure 6: Bandwidth matrices of optimal, individual smoothing parameters for the two-sided (top) and one-sided (bottom) nonparametric covariance estimation in regression model (2) regarding daily benchmark returns from 1999 to 2008. Gray highlighted elements designate manually adjusted bandwidths after CV.

The covariance estimates in figure 5 are depicted from April 1, 1999 (data point 68) due to reducing left boundary effects. Regarding the time-dependent, nonparametric covariance shape of *EquNA-RateUSD* some aspects from the empirical estimation recur in an impressive manner (compare figure 1 and attend the multiple scale). From spring 2007 to March 2008 there is a significant increase from a zero-level into the positive area of covariances. A more drastic upward jump accompanies the collapse of Lehman Brothers in September 2008. The one-sided estimate lags visually behind its two-sided 'foreseeing' counterpart; bandwidths  $h^{opt} = 20$  and  $h_{(1)}^{opt} = 70$  (originally from CV) enable a smooth but heteroscedastic course. The 258-day empirical covariance is inflexible and obviously behind the return dynamics and the nonparametric dimensions. As another example, the covariance *EquNA-CurrUSD* moves in the opposite direction in fall 2008.

It turns from a long lasting zero covariance into a negative interdependence, again with impressive peaks of the two-sided ( $h^{\text{opt}} = 45$ ) and the one-sided nonparametric estimates ( $h_{(1)}^{\text{opt}} = 48$ ). The *RateUSD-CurrUSD* covariance is volatile around zero with bigger amplitudes in the economic crisis from 2002 to 2004 and the biggest (negative) amplitude at the end of 2008.<sup>19</sup> As already observed for empirical estimators, large jumps of the covariance between *CredSta* and the equity-, interest rate- and currency example occurred mainly in 2008, but with a multiple in values for the nonparametric approach. Especially for *EquNA-CredSta* there is a pretty smooth course until 2007, before the graph explodes in both directions within the last 12 months. This is partially caused by small optimal bandwidths  $h^{\text{opt}} = 23$  and  $h_{(1)}^{\text{opt}} = 25$ , where the latter is manually adjusted. The matrices of all optimal, individual bandwidths regarding the two-sided and one-sided PNWEs are provided in figure 6. As mentioned before, bandwidth adjustments with reference to volatility estimates are necessary in almost half of the series and are visualized in gray.<sup>20</sup>

Parameters for complete implementation of the multivariate non-stationary model										
Two-sided setup	EquEUR	EquexEUR	EquNA	EquAsP	EquEM	RateEUR	RateUSD	RateJPY	CredSta	CredSwa
=> NWE volatility:										
CV-optimal BW	23	21	23	17	22	123	28	24	8	38
=> innovation fitting:										
m-	n.a.	n.a.	n.a.	n.a.	n.a.	11.1232	41.2850	36.6018	n.a.	17.7202
c-	n.a.	n.a.	n.a.	n.a.	n.a.	4.1727	9.2963	7.4048	n.a.	5.6140
m+	n.a.	n.a.	n.a.	n.a.	15.8238	5.1683	11.2614	n.a.	n.a.	9.7926
c+	n.a.	n.a.	n.a.	n.a.	4.4073	2.7840	4.1451	n.a.	n.a.	4.0188
One-sided setup										
=> NWE volatility:										
CV-optimal BW	24	21	23	16	20	116	111	27	25	117
=> innovation fitting:										
m-	n.a.	n.a.	n.a.	n.a.	n.a.	11.5698	7.8002	n.a.	13.1045	7.3833
c-	n.a.	n.a.	n.a.	n.a.	n.a.	4.3754	3.5921	n.a.	4.7613	3.4742
m+	n.a.	n.a.	n.a.	n.a.	n.a.	5.2445	6.5855	26.6930	6.1066	5.6803
c+	n.a.	n.a.	n.a.	n.a.	n.a.	2.8633	3.1978	7.1152	2.9736	2.9262
Two-sided setup										
=> NWE volatility:										
CV-optimal BW	26	32	20	18	19	15		19	20	22
=> innovation fitting:										
m-	1142.5550	19.3434	15.7418	12.6562	5.7613	13.8465		n.a.	n.a.	n.a.
c-	46.9752	5.9009	5.1527	4.6956	2.5574	4.2185		n.a.	n.a.	n.a.
m+	8.1223	6.2152	424.1662	5.9961	21.6878	n.a.		n.a.	34.2009	n.a.
c+	3.5217	2.9989	28.8808	2.8354	6.8743	n.a.		n.a.	8.0329	n.a.
One-sided setup										
=> NWE volatility:										
CV-optimal BW	77	88	21	27	15	17		34	21	29
=> innovation fitting:										
m-	7.3871	7.0479	n.a.	12.5347	n.a.	n.a.		76.5546	n.a.	n.a.
c-	3.5181	3.3783	n.a.	4.4796	n.a.	n.a.		12.5851	n.a.	n.a.
m+	4.2712	3.8663	n.a.	10.5799	n.a.	n.a.		n.a.	n.a.	n.a.
c+	2.3111	2.1622	n.a.	4.3305	n.a.	n.a.		n.a.	n.a.	n.a.
Two-sided setup										
=> NWE volatility:										
CV-optimal BW	28	14	92	78	16	8	34	50	77	59
=> innovation fitting:										
m-	6.7317	n.a.	7.7889	7.3267	n.a.	n.a.	9.2226	52.7573	201.2165	32.8947
c-	3.2483	n.a.	3.8243	3.7445	n.a.	n.a.	4.3181	10.7170	21.4096	8.2616
m+	4.6704	n.a.	13.9263	32.1331	n.a.	n.a.	8.1475	13.2922	29.3699	47.8209
c+	2.3996	n.a.	4.5718	7.0076	n.a.	n.a.	3.1521	4.5188	6.9351	9.0761
One-sided setup										
=> NWE volatility:										
CV-optimal BW	45	68	86	70	19	19	37	97	66	58
=> innovation fitting:										
m-	4.9542	5.3422	7.9036	7.4133	n.a.	n.a.	11.1896	13.5296	77.9578	31.0653
c-	2.5830	2.9276	3.6996	3.5424	n.a.	n.a.	4.5557	4.9784	12.6287	7.7819
m+	4.6770	16.2650	16.5580	n.a.	n.a.	n.a.	20.6648	10.1071	32.2023	412.8480
c+	2.5395	5.1640	5.3304	n.a.	n.a.	n.a.	5.8140	4.1037	7.6584	27.9816

Figure 7: Parameters for the non-stationary modeling (2) of daily benchmark returns from 1999 to 2008, based on individual smoothing of covariance matrix elements and with reference to the two-sided and the one-sided implementation. Gray highlighted elements designate manually adjusted bandwidths after CV. Cells with entry 'n.a.' identify series where the Pearson VII innovation fitting failed.

<sup>19</sup>The (counter) dynamics within the last few days of 2008 might be influenced through boundary effects at all examples.

<sup>20</sup>The numeraire *CurrEUR* (constant 1) is excluded from nonparametric regression and the model parametrization hereafter.

The last step of modeling the return dynamics is to estimate the series of realized innovations  $\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n$ , that is done independently in its components by the ratio of centered returns  $R_{k,t}$  and volatility estimates  $\hat{\sigma}_k(t)$  (or  $\tilde{R}_{k,t}/\hat{\sigma}_{(1)k}(t)$ , respectively), and to fit the asymmetric Pearson type VII density  $f_{m+,c+,m-,c-}^{\text{VII}}$  univariate for dimensions  $k = 1, \dots, 30$ . The complete parametrization of the non-stationary approach regarding the 10-year daily return series and the individual smoothing conception is presented in figure 7, distinguished between the two-sided and the one-sided setup. As seen on the diagonals of bandwidth matrices before, only for three benchmarks the (two-sided) bandwidths of volatility estimation have to be adjusted manually. But the drawback of individual smoothing is a failure in the innovation fitting to a certain extent (44% of all tail cases), because a kurtosis lower 3 results. It is noticeable that the Pearson VII approximation falls through many times when the volatility estimates are based on small (CV-optimal) bandwidths and NWEs entail already most of return dynamics, which implies an innovation distribution with weaker tails than the Gaussian. Otherwise, not perfectly (not individual) calibrated, oversmoothed volatility estimates on the same series could produce heavier tailed innovations and the complete non-stationary framework (2) holds. Hence, there might be a trade-off between the quality in volatility estimation and a successful fit of innovations. Gürtler and Rauh (2009) comment on this effect more detailed. At the successful residual approximations, we observe indeed asymmetries between the tails (or only one Pearson VII tail exists) and some heavy-tailed examples, e.g. *RateEUR* (right tail = loss tail), *CredAA* or *CurrCAD* (left tail = loss tail).<sup>21</sup>

Coming back to the risk interdependencies between exposure classes, we are interested on the heteroscedasticity of a standardized measure. For the time being we continue with individual smoothing. According to Pearson correlations, we apply our nonparametric (co)variance estimates to gain a first definition of nonparametric correlations:

$$\hat{\rho}_{(\cdot)k,l}(t) = \frac{\hat{\sigma}_{(\cdot)k,l}^2(t)}{\hat{\sigma}_{(\cdot)k}(t) \hat{\sigma}_{(\cdot)l}(t)}. \quad (24)$$

Although its components are (asymptotically) consistent, the fraction is not as there is no functional relation between the corresponding bandwidths  $h_{k,l}$ ,  $h_k$  and  $h_l$ . The different degrees of smoothing might even produce correlation estimators that are absolutely greater than one. Even if no imperfect components are included, the corresponding correlation matrix  $\hat{\varrho}_{(\cdot)h}(t)$  might be not positive (semi)definite due to individual smoothing (bandwidths from matrix  $h$ ). This is unacceptable also in practice for the task of calculating portfolio variances (implying portfolio mixes with a variance lower zero). Despite these imperfections, the dynamics of correlation estimates (24) are examined in figure 8; outliers from the interval  $[-1, 1]$  are signed with a warning triangle.<sup>22</sup> Regarding the standard examples, we recognize on the one hand main patterns of the empirical correlations (compare figure 2) and some directions of the PNWEs from figure 5. On the other hand significant changes of correlation estimates over time are observed,<sup>23</sup> although the upper four illustrated examples tend to undersmoothing.

For example, the *EquNA-RateUSD* correlation is estimated nonparametrically around  $-0.4$  in fall 1999 and increases significantly in the following years, where a level of about 0.6 is relatively stable from sum-

<sup>21</sup>The lower  $m$ - and  $c$ -, the heavier the tail. The measure  $2m - 1$  (Student's df) is called tail index point estimate.

<sup>22</sup>The estimates start again at April 1, 1999 to reduce left boundary effects.

<sup>23</sup>Significance in a statistical sense is to be concluded with a transition of the  $z$ -test from section 2.

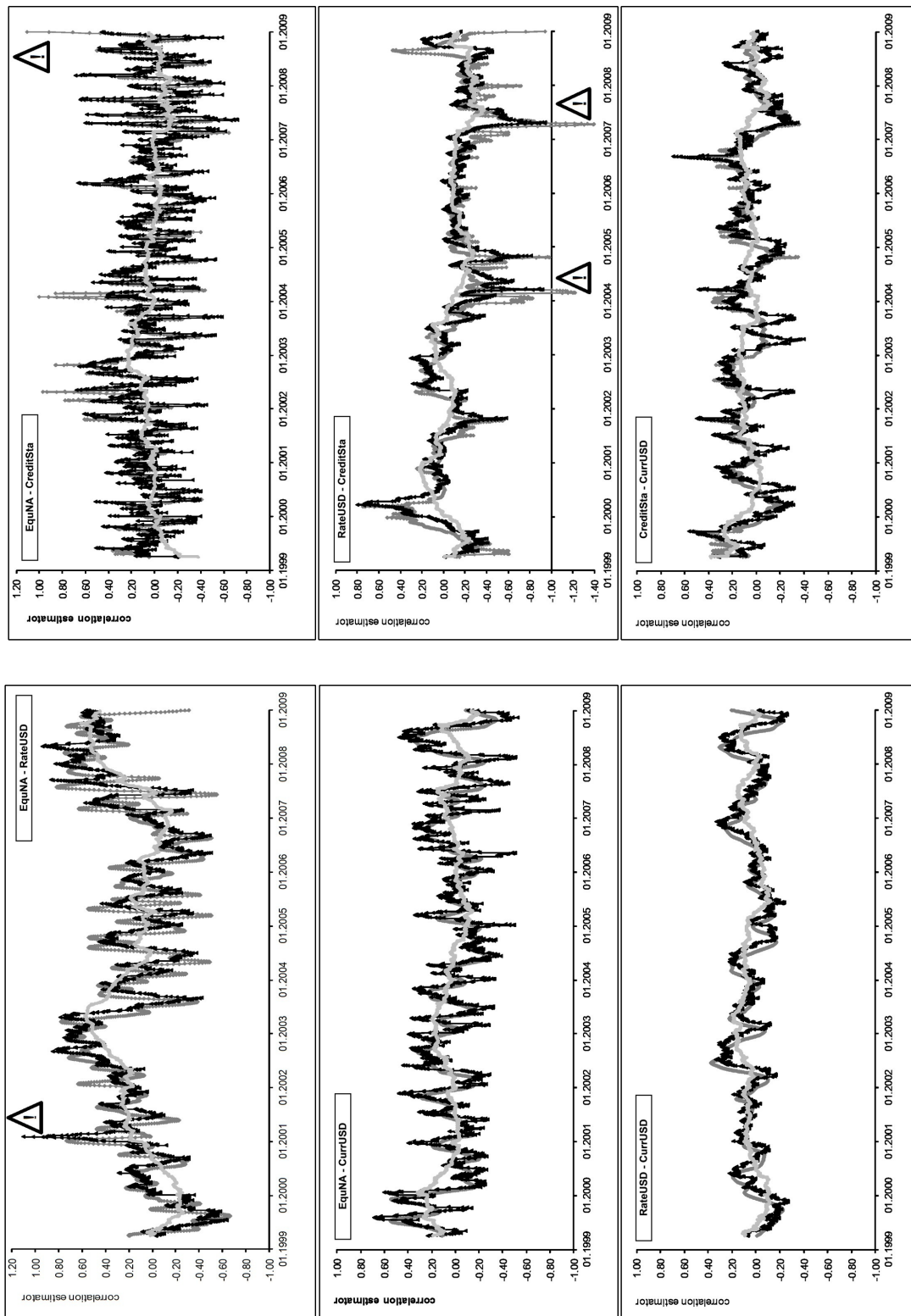


Figure 8: Correlation estimates regarding daily returns of the pairs of series *EquiNA*, *RateUSD*, *CreditSta* and *CurrUSD*. From the individual smoothing of regression model (2) the two-sided standardized PNWEs are depicted in dark gray bullets and the one-sided standardized PNWEs in black triangles. The 258-day empirical correlation (bright gray line) is opposed each with.

mer 2002 to spring 2003. At the following decline it is visible how the one-sided estimate lags behind the two-sided estimator (e.g. on May 15, 2003:  $\hat{\rho} = 0.09$  against  $\hat{\rho}_{(1)} = 0.67$ ). After volatile years around zero, the correlation becomes significantly positive from summer 2007. Again the *EquNA-CurrUSD* dynamics are converse, starting with a nonparametric correlation of about 0.5 in November 1999, interim lows of approximately  $-0.2$  in August 2000 and May 2003 and e.g. another upward jump from January to June 2008 (from  $-0.2$  to  $0.4$ ). Most oscillating is the correlation graph of *EquNA-CredSta* due to bandwidths lower than or equal 25 days in all components of the approach (24). The smoothest shape deliver the estimates regarding *RateUSD-CurrUSD*, but substantial differences of correlations are also observed between November 15, 1999 ( $\hat{\rho} = -0.10$ ,  $\hat{\rho}_{(1)} = -0.24$ ) and May 15, 2003 ( $\hat{\rho} = 0.15$ ,  $\hat{\rho}_{(1)} = 0.21$ ).

We used the preceding two dates as representative points for comparing the differences of correlation matrices  $\hat{\rho}_{(\cdot)h}(t)$  in time. Figures 9 and 10 present each the two-sided and the one-sided nonparametric setup (on lower triangular) against their empirical estimates (on upper triangular) and calculate the differences between November 15, 1999 and May 15, 2003. Contradictions to the correlation measure ( $|\hat{\rho}_{(\cdot)k,l}(t)| > 1$ ) are highlighted gray. Estimator deviations over time with a 5% level of significance are also highlighted (conservative  $z$ -test adjusted to average two-/ one-sided bandwidths).<sup>24</sup> In terms of the two-sided estimators about 30% of the realizations differ significantly, where the average absolute deviation of correlations is 0.25. About 23% of the time differences of one-sided estimates are significant (av. abs. dev. 0.26) and even 25% of empirical correlations (av. abs. dev. 0.13). Hence, the fraction of time-dependent correlations (exemplified at the two points in time) has a similar dimension for the three estimates, and of course the nonparametric approaches vary absolutely strongest. By comparing the lower to the upper triangle the differences (at the same time point) between nonparametric and empirical estimates become obvious. For instance the aforementioned correlation *EquNA-RateUSD* is estimated as 0.09 two-sided nonparametrically, but still 0.56 empirically on March 15, 2003 (middle tableau in figure 9). Another example is the distance of *EquNA-CurrUSD* correlation estimators at the same date, where the one-sided standardized PNWE  $-0.20$  differs significantly from the empirical correlation to the amount of 0.15 (middle tableau in figure 10).

Moreover, a contradiction to the definiteness requirements of covariance and correlation matrices arises from the nonparametric estimates of figures 9 and 10.<sup>25</sup> On November 15, 1999 the matrix of two-sided correlation estimates has six negative eigenvalues, while the one-sided correlation matrix has even eight. On May 15, 2003 nonparametric correlation matrices have seven respectively eight negative eigenvalues and, hence, are not positive definite again. The effect is strongly pronounced due to pairwise correlations lower than  $-1$ . Even in the case of (coincidentally) well defined individual correlations (i.e. all  $|\hat{\rho}_{(\cdot)k,l}(t)| \leq 1$ ), the matrices need not to be positive semidefinite. Figure 11 presents an example for appointed date June 30, 2006: A mixed EUR-funds invests 1,000,000 EUR with 10% in global equities (currency unhedged) and 90% in fixed-income (bonds and linear derivatives: treasuries, corporates, high yields, emerging markets

<sup>24</sup>More deviating values are obvious between November 1999 and April 2008, but also between some other points within the series. We chose to compare preferably close time points. Matrices before November 1999 and after April 2008 should not be analyzed due to boundary effects.

<sup>25</sup>Due to the role of *CurrEUR*, having a zero variance and zero covariances to the other (stochastic) exposure classes, the covariance matrix cannot achieve positive definiteness, but should be positive semidefinite. The correlation matrix is constructed with unit vectors in the line and column corresponding to *CurrEUR*, and hence is required to be positive definite.

15.11.99 empirical estimates

Corr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Sva	Credit AAA	Credit AA	Credit A	Credit BBB	Credit EM	Credit HY	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM	
EquEUR	1.00	0.86	0.45	0.33	0.50	-0.18	-0.09	0.03	0.17	0.09	0.10	0.14	0.12	0.12	-0.42	0.07	0	0.22	-0.26	0.27	0.04	0.31	0.40	0.41	0.08	0.32	0.21	0.30	0.28	0.36	
EquexEUR	0.82	1.00	0.41	0.34	0.48	-0.10	-0.04	-0.01	0.13	0.00	0.01	0.05	0.01	0.03	-0.38	0.02	0	0.21	-0.17	0.24	0.01	0.22	0.32	0.31	0.01	0.24	0.14	0.26	0.24	0.33	
EquNA	0.58	0.62	1.00	0.08	0.32	-0.22	-0.24	0.00	-0.01	0.08	0.04	0.07	0.07	0.09	-0.31	0.21	0	0.15	-0.06	0.08	0.07	0.10	0.24	0.13	0.06	0.13	0.06	0.32	0.39	0.43	
EquASP	0.23	0.10	0.37	1.00	0.45	-0.01	-0.03	0.01	0.17	0.02	0.02	0.04	0.06	0.05	-0.08	0.00	0	0.10	-0.23	0.17	0.03	0.19	0.18	0.26	0.02	0.12	0.12	0.06	0.00	-0.01	
EquEM	0.47	0.58	0.50	0.24	1.00	0.03	-0.02	0.01	0.07	-0.13	-0.09	-0.02	-0.05	-0.07	-0.52	-0.06	0	0.12	-0.40	0.26	0.06	0.36	0.25	0.33	0.10	0.26	0.22	0.22	0.23	0.33	
RateEUR	-0.10	-0.04	-0.33	0.18	-0.11	1.00	0.38	0.11	-0.21	-0.62	-0.41	-0.34	-0.31	-0.32	0.03	-0.40	0	0.00	-0.04	-0.01	0.04	0.07	-0.01	0.00	0.06	0.06	0.06	0.06	-0.09	-0.07	-0.11
RateUSD	-0.19	-0.16	-0.30	0.13	-0.22	0.51	1.00	0.09	-0.13	-0.22	-0.09	-0.12	-0.12	-0.06	-0.15	-0.82	0	-0.10	-0.07	-0.05	-0.08	0.02	-0.11	-0.06	-0.11	0.01	-0.05	-0.01	-0.03	0.00	
RateJPY	-0.02	-0.06	0.08	0.28	-0.04	0.22	0.17	1.00	0.09	-0.10	-0.03	-0.10	-0.07	-0.09	-0.06	-0.04	0	-0.02	0.01	0.06	0.11	0.03	0.02	0.03	0.21	0.12	0.10	-0.03	-0.03	0.02	
CreditSta	0.04	0.00	-0.16	0.21	0.07	0.30	0.25	0.06	1.00	0.27	-0.10	-0.13	-0.13	-0.22	-0.07	0.15	0	0.20	-0.09	0.14	0.03	0.13	0.24	0.27	0.13	0.16	0.16	0.01	0.02	0.03	
CreditSva	-0.02	-0.08	0.39	-0.01	-0.03	-0.67	-0.25	-0.10	-0.77	1.00	0.50	0.43	0.47	0.35	0.03	0.28	0	0.06	0.10	0.03	-0.06	0.02	0.07	0.01	0.02	0.00	-0.02	0.01	0.02	0.03	
CreditAAA	-0.17	-0.15	0.00	-0.08	-0.12	-0.47	-0.05	0.00	-0.70	0.58	1.00	0.90	0.89	0.79	0.03	0.12	0	0.01	0.06	0.08	-0.06	0.04	0.00	0.01	-0.04	0.03	-0.01	0.00	-0.01	-0.03	
CreditAA	-0.07	0.05	-0.03	-0.16	-0.05	-0.43	-0.18	-0.06	-0.65	0.53	1.63	1.00	0.87	0.73	0.02	0.14	0	0.00	0.02	0.12	-0.04	0.05	0.02	0.02	-0.06	0.05	0.00	0.02	0.01	0.01	
CreditA	-0.09	-0.10	-0.02	-0.18	-0.20	-0.40	0.09	-0.13	-0.39	0.57	0.68	0.83	1.00	0.74	0.00	0.13	0	0.02	0.04	0.06	-0.05	0.03	0.02	-0.01	-0.04	0.00	0.02	0.02	0.01		
CreditBBB	0.01	0.01	0.20	-0.12	-0.04	-0.39	-0.09	-0.16	-0.58	0.51	0.82	0.48	0.47	1.00	0.01	0.08	0	0.02	0.05	0.01	-0.06	-0.01	-0.02	-0.04	-0.10	0.05	0.00	0.00	-0.01	-0.02	
CreditEM	-0.47	-0.54	-0.54	-0.24	-0.42	-0.01	0.04	0.03	0.20	-0.12	-0.11	-0.01	-0.20	-0.18	1.00	1.00	0	-0.11	0.29	-0.15	-0.07	-0.31	-0.18	-0.21	-0.16	-0.19	-0.19	-0.24	-0.22	-0.41	
CreditHY	0.13	0.05	0.26	-0.18	-0.10	-0.52	-1.41	-0.23	-0.21	0.31	0.30	0.18	0.00	0.14	0.03	0.15	0	0.08	0.10	-0.04	0.02	-0.11	0.11	0.08	0.03	-0.04	-0.04	-0.01	0.01	-0.03	
CurrEUR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	
CurrGBP	0.27	0.11	0.24	0.18	0.31	0.00	-0.08	-0.12	0.00	-0.02	-0.32	-0.18	-0.35	-0.07	-0.30	0.12	0	1.00	-0.07	0.28	0.14	0.35	0.67	0.55	0.38	0.51	0.49	0.27	0.31	0.25	
CurrCHF	-0.11	-0.17	-0.16	-0.22	-0.47	-0.03	0.16	0.08	-0.24	0.25	0.01	0.06	0.15	0.19	0.11	-0.10	0	-0.23	1.00	-0.17	-0.02	-0.20	-0.17	-0.29	-0.08	-0.17	-0.19	-0.13	-0.08	-0.09	
CurrSEK	0.33	0.20	0.02	0.20	0.28	0.03	-0.28	0.14	0.11	-0.09	-0.26	-0.14	-0.31	-0.25	-0.27	-0.01	0	0.33	-0.27	1.00	0.21	0.65	0.42	0.43	0.24	0.41	0.33	0.11	0.18	0.14	
CurrDKK	-0.26	-0.34	0.09	0.02	0.20	0.08	0.04	0.19	-0.27	-0.01	-0.07	-0.08	-0.10	-0.06	-0.18	0.07	0	0.28	-0.07	1.00	0.14	1.00	0.15	0.17	0.15	0.16	0.14	0.05	0.10	0.11	
CurrNok	0.53	0.46	0.13	0.14	0.45	0.15	0.01	0.33	0.23	-0.21	-0.32	-0.29	-0.41	-0.15	-0.41	-0.02	0	0.78	-0.08	0.60	1.10	1.00	0.52	0.55	0.32	0.50	0.45	0.18	0.23	0.24	
CurrUSD	0.29	0.29	0.41	0.23	0.25	-0.04	-0.10	-0.01	0.00	-0.01	-0.05	-0.05	-0.08	-0.06	-0.21	0.27	0	0.63	-0.25	0.44	0.27	0.52	1.00	0.83	0.44	0.66	0.57	0.28	0.34	0.29	
CurrCAD	0.36	0.39	0.29	0.34	0.39	0.00	-0.13	0.03	0.06	0.03	-0.04	-0.04	-0.08	-0.11	-0.35	0.08	0	0.67	-0.40	0.63	0.25	1.00	0.78	1.00	0.35	0.74	0.65	0.32	0.35	0.34	
CurrJPY	0.01	0.03	-0.07	0.08	0.10	0.11	0.06	0.02	0.04	-0.02	-0.15	-0.24	-0.28	-0.31	-0.06	-0.04	0	0.47	-0.09	0.38	0.21	0.69	0.88	0.66	1.00	0.35	0.36	0.22	0.14	0.19	
CurrAUD	0.12	-0.13	0.08	0.09	0.17	0.14	-0.08	0.02	0.09	-0.12	-0.43	-0.23	-0.38	-0.11	-0.19	0.43	0	0.06	-0.14	0.62	0.24	0.42	0.74	0.92	0.55	1.00	0.82	0.21	0.24	0.25	
CurrNZD	0.26	0.29	0.13	0.10	0.18	0.13	-0.12	0.00	0.13	-0.16	-0.29	-0.02	-0.06	-0.03	-0.11	0.53	0	0.70	-0.19	0.44	0.17	0.43	0.60	0.71	0.53	0.78	1.00	0.12	0.15	0.14	
CurrSGD	0.25	0.31	0.41	0.06	0.19	-0.12	-0.05	0.02	-0.20	0.18	-0.01	-0.03	-0.03	-0.05	-0.40	0.09	0	0.48	-0.16	0.09	0.00	0.29	0.40	0.37	0.37	0.19	1.00	0.84	0.82	0.84	
CurrHKD	0.15	0.31	0.37	0.00	0.23	-0.10	-0.03	-0.03	-0.13	0.09	0.01	0.00	-0.02	-0.07	-0.44	-0.03	0	0.51	-0.24	0.13	0.04	0.30	0.47	0.41	0.44	0.27	0.19	0.93	1.00	0.89	
CurrEM	0.23	0.38	0.46	0.04	0.30	-0.13	-0.03	-0.01	-0.15	0.09	-0.01	-0.02	-0.02	-0.08	-0.48	0.04	0	0.52	-0.25	0.13	0.03	0.31	0.42	0.39	0.34	0.24	0.21	0.97	1.00	1.00	

15.05.03 empirical estimates

Corr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Sva	Credit AAA	Credit AA	Credit A	Credit BBB	Credit EM	Credit HY	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM
EquEUR	1.00	0.93	0.61	0.23	0.45	0.66	0.46	0.10	0.26	-0.21	-0.12	-0.29	-0.39	-0.46	-0.45	-0.50	0	0.13	-0.24	0.36	0.03	0.16	0.31	0.37	0.00	0.27	0.20	0.25	0.30	0.35
EquexEUR	0.90	1.00	0.51	0.26	0.47	0.56	0.38	0.11	0.26	-0.11	-0.05	-0.23	-0.31	-0.43	-0.36	-0.41	0	0.12	-0.24	0.41	0.03	0.16	0.33	0.40	-0.01	0.30	0.22	0.19	0.25	0.29
EquNA	0.71	0.61	1.00	0.11	0.29	0.49	0.56	0.13	0.14	-0.26	-0.11	-0.18	-0.27	-0.21	-0.44	-0.56	0	0.05	-0.07	0.14	0.03	0.07	0.15	0.23	0.04	0.11	0.05	0.36	0.40	0.39
EquASP	0.32	0.39	0.06	1.00	0.60	0.13	0.06	0.05	0.16	0.24	0.08	-0.07	-0.17	-0.35	-0.16	-0.17	0	0.03	-0.23	0.24	-0.08	0.07	0.18	0.22	0.01	0.27	0.23	-0.01	-0.02	0.03
EquEM	0.47	0.49	0.36	0.51	1.00	0.27	0.23	0.09	0.22	0.19	0.16	-0.07	-0.23	-0.44	-0.38	-0.35	0	0.13	-0.38	0.25	-0.03	0.10	0.32	0.36	0.10	0.35	0.30	0.09	0.07	0.15
RateEUR	0.65	0.54	0.33	0.11	0.28	1.00	0.57	0.11	0.28	-0.60	-0.35	-0.46	-0.52	-0.46	-0.33	-0.54	0	0.15	-0.13	0.24	0.04	0.09	0.27	0.30	0.06	0.17	0.17	0.34	0.40	0.39
RateUSD	-0.08	0.32	0.09	0.37	0.12	0.71	1.00	0.14	0.06	-0.29	-0.15	-0.20	-0.28	-0.24	-0.46	-0.64	0	0.09	-0.16	0.11	0.10	0.01	0.19	0.18	0.04	0.07	0.06	0.32	0.35	0.34
RateJPY	-0.20	-0.18	-0.08	1.46	0.10	0.69	0.43	1.00	0.03	-0.04	0.06	0.00	-0.11	-0.15	-0.08	-0.15	0	0.08	0.03											



15.11.99 empirical estimates

Corr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Sva	Credit AAA	Credit AA	Credit A	Credit BBB	Credit EM	Credit HY	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM	
EquEUR	1.00	0.86	0.45	0.33	0.50	-0.18	-0.09	0.03	0.17	0.09	0.10	0.14	0.12	0.12	-0.42	0.07	0	0.22	-0.26	0.27	0.04	0.31	0.40	0.41	0.08	0.32	0.21	0.30	0.28	0.36	
EquexEUR	0.87	1.00	0.41	0.34	0.48	-0.10	-0.04	-0.01	0.13	0.00	0.01	0.05	0.01	0.03	-0.38	0.02	0	0.21	-0.17	0.24	0.01	0.22	0.32	0.31	0.01	0.24	0.14	0.26	0.24	0.33	
EquNA	0.76	0.77	1.00	0.08	0.32	-0.22	-0.24	0.00	-0.01	0.08	0.04	0.07	0.07	0.09	-0.31	0.21	0	0.15	-0.06	0.08	0.07	0.10	0.24	0.13	0.06	0.13	0.06	0.32	0.39	0.43	
EquASP	0.26	0.22	0.30	1.00	0.45	-0.01	-0.03	0.01	0.17	0.02	0.02	0.04	0.06	0.05	-0.08	0.00	0	0.10	-0.23	0.17	0.03	0.19	0.18	0.26	0.02	0.12	0.12	0.06	0.32	-0.05	-0.01
EquEM	0.64	0.78	0.50	0.18	1.00	0.03	-0.02	0.01	0.07	-0.13	-0.09	-0.02	-0.05	-0.07	-0.52	-0.06	0	0.12	-0.40	0.26	0.06	0.36	0.25	0.33	0.10	0.26	0.22	0.22	0.23	0.33	
RateEUR	-0.25	-0.14	-0.45	0.09	-0.08	1.00	0.38	0.11	-0.21	-0.62	-0.41	-0.34	-0.31	-0.32	0.03	-0.40	0	0.00	-0.04	-0.01	0.04	0.07	-0.01	0.00	0.06	0.06	0.06	0.06	-0.09	-0.07	-0.11
RateUSD	-0.28	-0.21	-0.29	0.01	-0.23	0.37	1.00	0.09	-0.13	-0.22	-0.09	-0.12	-0.12	-0.06	-0.15	-0.82	0	-0.10	-0.07	-0.05	-0.08	0.02	-0.11	-0.06	-0.11	0.01	-0.05	-0.01	-0.03	0.00	
RateJPY	0.03	0.04	0.03	0.23	0.43	0.15	0.15	1.00	0.09	0.10	-0.03	-0.10	-0.07	-0.09	-0.06	-0.04	0	-0.02	0.01	0.06	0.11	0.03	0.02	0.03	0.21	0.12	0.10	-0.03	-0.03	0.02	
CreditSta	0.09	0.04	-0.10	0.21	0.32	0.13	-0.02	0.29	1.00	0.27	-0.10	-0.13	-0.22	-0.07	0.15	0	0	0.20	-0.09	0.14	0.03	0.13	0.24	0.27	0.13	0.16	0.16	0.01	0.02	0.03	
CreditSva	0.07	-0.10	0.12	0.07	-0.19	-0.62	-0.19	0.18	-0.14	1.00	0.50	0.43	0.47	0.35	0.03	0.28	0	0.06	0.10	0.03	-0.06	0.02	0.07	0.01	0.02	0.00	-0.02	0.01	0.02	0.03	
CreditAAA	-0.05	-0.13	-0.17	0.03	-0.26	-0.41	-0.13	0.01	-0.50	0.48	1.00	0.90	0.89	0.79	0.03	0.12	0	0.01	0.06	0.08	-0.06	0.04	0.00	0.01	-0.04	0.03	-0.01	0.00	-0.01	-0.03	
CreditAA	0.09	-0.04	-0.10	-0.05	-0.09	-0.29	-0.20	0.13	-0.52	0.33	0.91	1.00	0.87	0.73	0.02	0.14	0	0.00	0.02	0.12	-0.04	0.05	0.02	0.02	-0.06	0.05	0.00	0.02	0.01	0.01	
CreditA	0.11	-0.10	0.07	0.00	-0.16	-0.32	-0.15	0.06	-0.76	0.64	0.95	0.89	1.00	0.74	0.00	0.13	0	0.02	0.04	0.06	-0.05	0.03	0.02	-0.01	-0.04	0.00	0.02	0.02	0.01	0.01	
CreditBBB	0.28	0.14	0.06	-0.06	-0.42	-0.53	-0.21	-0.14	-0.73	0.78	1.01	0.96	0.75	1.00	0.01	0.08	0	0.02	0.05	0.01	-0.06	-0.01	-0.02	-0.04	-0.10	0.05	0.00	0.00	-0.01	-0.02	
CreditEM	-0.60	-0.69	-0.58	-0.14	-0.47	0.15	0.00	-0.01	0.13	0.03	-0.04	-0.17	-0.34	-0.40	1.00	1.05	0	-0.11	0.29	-0.15	-0.07	-0.31	-0.18	-0.21	-0.16	-0.19	-0.19	-0.24	-0.22	-0.41	
CreditHY	0.41	0.31	0.46	0.39	0.09	-0.49	-1.07	-0.22	-0.08	0.27	0.13	0.19	0.40	0.35	-0.26	1.00	0	0.08	0.10	-0.04	0.02	-0.11	0.11	0.00	0.03	-0.04	-0.04	-0.01	0.01	-0.03	
CurrEUR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
CurrGBP	0.17	0.10	0.22	0.24	-0.03	0.03	-0.21	-0.24	0.15	-0.07	-0.12	-0.14	-0.15	-0.03	-0.27	0.28	0	1.00	-0.07	0.28	0.14	0.35	0.67	0.55	0.38	0.51	0.49	0.27	0.31	0.25	
CurrCHF	-0.34	-0.43	-0.43	-0.38	-0.40	-0.04	-0.01	0.01	-0.03	0.18	0.18	0.09	0.22	0.13	0.03	-0.28	0	-0.36	1.00	-0.17	-0.02	-0.20	-0.17	-0.29	-0.08	-0.17	-0.19	-0.13	-0.08	-0.09	
CurrSEK	0.37	0.32	0.15	0.33	0.34	0.04	-0.09	0.28	0.33	-0.07	-0.11	-0.03	-0.16	-0.15	-0.19	0.02	0	0.40	-0.33	1.00	0.21	0.65	0.42	0.43	0.24	0.41	0.33	0.11	0.15	0.14	
CurrDKK	-0.21	-0.12	-0.12	-0.02	-0.07	0.16	-0.23	0.12	0.04	-0.21	-0.15	-0.15	-0.17	-0.10	0.14	-0.06	0	0.20	0.19	0.22	1.00	0.14	0.15	0.17	0.15	0.16	0.14	0.05	0.10	0.11	
CurrNok	0.25	0.25	0.10	0.36	0.45	0.16	0.05	0.13	0.18	-0.06	-0.14	-0.13	-0.23	-0.20	-0.14	0.02	0	0.53	-0.34	0.66	1.11	1.00	0.52	0.55	0.32	0.50	0.45	0.18	0.23	0.24	
CurrUSD	0.38	0.39	0.50	0.32	0.12	-0.04	-0.24	0.01	0.14	-0.05	-0.07	-0.08	-0.07	-0.09	-0.17	0.67	0	0.48	-0.16	0.64	0.27	0.31	1.00	0.83	0.44	0.66	0.57	0.28	0.34	0.29	
CurrCAD	0.35	0.37	0.28	0.48	0.47	-0.02	-0.21	0.08	0.13	-0.08	-0.08	-0.05	-0.08	-0.15	-0.39	0.22	0	0.87	-0.48	0.69	0.21	0.65	1.00	1.00	0.35	0.74	0.65	0.32	0.35	0.34	
CurrJPY	0.04	-0.06	0.06	0.00	0.03	0.22	-0.41	0.41	0.48	0.01	-0.09	-0.09	-0.42	-0.25	0.15	0.39	0	0.92	0.05	0.54	0.24	0.40	0.98	0.79	1.00	0.35	0.36	0.22	0.14	0.19	
CurrAUD	0.17	-0.05	0.18	0.07	-0.10	0.16	-0.01	0.04	0.11	-0.15	-0.29	-0.13	-0.15	0.01	-0.25	0.30	0	0.86	-0.01	0.66	0.22	0.46	0.71	0.82	0.40	1.00	0.82	0.21	0.24	0.25	
CurrNZD	0.46	0.32	0.31	0.38	0.34	0.12	-0.12	0.06	0.24	-0.16	-0.17	-0.11	-0.10	-0.07	-0.24	0.37	0	0.74	-0.17	0.55	0.09	0.42	0.65	0.74	0.36	0.79	1.00	0.12	0.15	0.14	
CurrSGD	0.47	0.46	0.62	-0.02	0.33	-0.24	-0.03	-0.10	-0.05	0.03	-0.02	-0.03	0.02	-0.05	-0.45	0.12	0	0.35	-0.19	0.04	0.10	0.16	0.23	0.24	0.21	0.09	-0.02	1.00	0.84	0.82	
CurrHKD	0.24	0.36	0.41	-0.05	0.26	-0.25	-0.10	-0.25	-0.03	0.06	-0.08	-0.07	-0.04	-0.17	-0.68	0.12	0	0.37	-0.42	0.10	0.03	0.13	0.34	0.29	0.24	0.08	-0.01	0.91	1.00	0.89	
CurrEM	0.33	0.40	0.65	0.02	0.38	-0.28	-0.09	-0.19	-0.07	0.04	-0.06	0.07	-0.01	-0.16	-0.72	0.09	0	0.34	-0.43	0.13	0.00	0.23	0.26	0.29	0.21	0.08	0.02	0.95	1.05	1.00	

15.05.03 empirical estimates

Corr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Sva	Credit AAA	Credit AA	Credit A	Credit BBB	Credit EM	Credit HY	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM
EquEUR	1.00	0.93	0.61	0.23	0.45	0.66	0.46	0.10	0.26	-0.21	-0.12	-0.29	-0.39	-0.46	-0.45	-0.50	0	0.13	-0.24	0.36	0.03	0.16	0.31	0.37	0.00	0.27	0.20	0.25	0.30	0.35
EquexEUR	0.88	1.00	0.51	0.26	0.47	0.56	0.38	0.11	0.26	-0.11	-0.05	-0.23	-0.31	-0.43	-0.36	-0.41	0	0.12	-0.24	0.41	0.03	0.16	0.33	0.40	-0.01	0.30	0.22	0.19	0.25	0.29
EquNA	0.76	0.84	1.00	0.11	0.29	0.49	0.56	0.13	0.14	-0.26	-0.11	-0.18	-0.27	-0.21	-0.44	-0.56	0	0.05	-0.07	0.14	0.03	0.07	0.15	0.23	0.04	0.11	0.05	0.36	0.40	0.39
EquASP	0.21	0.26	0.23	1.00	0.60	0.13	0.06	0.05	0.16	0.24	0.08	-0.07	-0.17	-0.35	-0.16	-0.17	0	0.03	-0.23	0.24	-0.08	0.07	0.18	0.22	0.01	0.27	0.23	-0.01	-0.02	0.03
EquEM	0.49	0.53	0.49	0.31	1.00	0.27	0.23	0.09	0.22	0.19	0.16	-0.07	-0.23	-0.44	-0.38	-0.35	0	0.13	-0.38	0.25	-0.03	0.10	0.32	0.36	0.10	0.35	0.30	0.09	0.07	0.15
RateEUR	1.03	0.96	0.73	0.02	0.32	1.00	0.57	0.11	0.28	-0.60	-0.35	-0.46	-0.52	-0.46	-0.33	-0.54	0	0.15	-0.13	0.24	0.04	0.09	0.27	0.30	0.06	0.17	0.17	0.34	0.40	0.39
RateUSD	0.68	0.65	0.67	-0.16	0.17	0.61	1.00	0.14	0.06	-0.29	-0.15	-0.20	-0.28	-0.24	-0.46	-0.64	0	0.09	-0.16	0.11	0.10	0.01	0.19	0.18	0.04	0.07	0.06	0.32	0.35	0.34
RateJPY	0.12	0.28	0.06	0.11	0.48	0.24	0.05	1.00	0.03	-0.04	0.06	0.00	-0.1																	

bonds; average modified duration 10%; currency hedged). The exposures are provided in the bottom vector (interest rate- and credit spread exposures expressed as bpv), a special long-short credit allocation is applied. In terms of the two-sided nonparametric estimates of volatilities and correlations a portfolio variance lower zero results ( $-585, 299$ ; corresponding to a relative 'standard deviation' of  $-0.08\%$ ), which is obviously wrong. As mentioned above, the reason is individual smoothing of (co)variances.

np_Corr	Equ EUR	Equ exEUR	Equ NA	Equ AsP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Swa	Credit AAA	Credit AA	Credit A	Credit BBB	Credit BB	Credit EM	Credit HY	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM
EquEUR	1.00	0.98	0.62	0.52	0.77	0.14	0.18	0.39	0.48	0.19	-0.10	-0.21	-0.02	-0.05	-0.57	-0.10	0.00	0.00	-0.04	-0.37	0.08	-0.31	-0.13	-0.08	0.01	0.11	0.28	-0.06	0.03	-0.31	0.06
EquexEUR	0.98	1.00	0.67	0.48	0.82	0.15	-0.17	0.44	0.53	0.21	-0.06	-0.17	-0.05	-0.13	-0.63	-0.16	0.00	0.00	-0.08	-0.31	0.12	-0.43	-0.08	-0.06	0.04	0.06	0.16	-0.11	0.02	-0.30	0.11
EquNA	0.62	0.67	1.00	0.16	0.51	0.02	-0.09	0.29	0.00	0.22	0.18	0.18	0.09	0.00	-0.35	0.05	0.00	0.00	-0.23	-0.14	0.09	-0.18	-0.12	0.12	0.02	-0.11	-0.04	-0.09	-0.22	-0.37	-0.19
EquAsP	0.52	0.48	0.16	1.00	0.78	0.09	-0.19	0.26	0.44	0.05	-0.09	-0.07	-0.20	-0.22	-0.23	0.05	0.00	0.00	0.27	0.47	0.13	-0.38	-0.46	-0.41	-0.18	0.03	0.43	0.07	-0.19	-0.26	0.13
EquEM	0.77	0.82	0.51	0.78	1.00	0.18	-0.16	0.33	0.29	0.08	0.06	0.01	-0.10	-0.20	-0.51	-0.18	0.00	0.00	0.16	0.44	0.17	-0.31	-0.03	-0.44	-0.07	-0.13	-0.06	-0.10	-0.21	-0.45	0.15
RateEUR	0.14	0.15	0.02	0.09	0.18	1.00	0.67	0.54	-0.01	-0.59	-0.59	-0.56	-0.57	-0.56	-0.45	-0.88	0.00	0.00	-0.04	-0.16	0.14	0.00	0.03	-0.11	-0.04	0.04	0.03	0.07	0.10	0.12	0.18
RateUSD	-0.18	-0.17	-0.09	-0.19	-0.16	0.67	1.00	0.36	-0.17	-0.53	-0.46	-0.37	-0.32	-0.26	-0.31	-0.85	0.00	0.00	0.05	0.18	-0.04	-0.02	0.07	0.05	0.09	-0.10	-0.16	0.09	0.22	0.37	0.27
RateJPY	0.39	0.44	0.29	0.26	0.33	0.54	0.36	1.00	0.40	-0.17	-0.37	-0.26	-0.33	-0.42	-0.43	-0.49	0.00	0.00	-0.19	0.25	-0.15	-0.19	0.11	0.08	0.07	-0.35	0.00	0.05	0.04	0.15	0.07
CreditSta	0.48	0.53	0.00	0.44	0.29	-0.01	-0.17	0.40	1.00	0.25	-0.35	-0.40	-0.12	-0.29	-0.07	-0.25	0.00	0.00	0.15	0.00	0.07	-0.14	0.13	0.32	0.19	0.38	0.40	0.26	0.13	-0.04	0.07
CreditSwa	0.19	0.21	0.22	0.05	0.08	-0.59	-0.53	-0.17	0.25	1.00	0.77	0.69	0.79	0.74	0.25	0.48	0.00	0.00	-0.29	-0.05	-0.03	0.10	0.30	0.03	-0.24	-0.02	-0.18	-0.21	-0.27	-0.33	
CreditAAA	-0.10	-0.06	0.18	-0.09	0.06	-0.59	-0.46	-0.37	-0.35	0.77	1.00	0.65	0.66	0.89	0.29	0.48	0.00	0.00	-0.15	-0.19	-0.03	0.11	0.07	0.21	0.06	0.00	0.00	-0.22	-0.31	-0.31	-0.42
CreditAA	-0.21	-0.17	0.18	-0.07	0.01	-0.56	-0.37	-0.26	-0.40	0.69	0.65	1.00	0.87	0.72	0.34	0.58	0.00	0.00	-0.15	-0.12	-0.08	0.11	-0.05	0.14	0.03	-0.31	-0.13	-0.18	-0.34	-0.29	-0.42
CreditA	-0.02	-0.05	0.09	-0.20	-0.10	-0.57	-0.32	-0.33	-0.12	0.79	0.66	0.87	1.00	0.89	0.47	0.56	0.00	0.00	-0.18	-0.15	0.00	0.14	0.16	0.21	-0.01	-0.25	-0.10	-0.23	-0.23	-0.22	-0.38
CreditBBB	-0.05	-0.13	0.00	-0.22	-0.20	-0.56	-0.26	-0.42	-0.29	0.74	0.89	0.72	0.89	1.00	0.64	0.65	0.00	0.00	-0.13	-0.08	-0.12	0.18	0.25	0.26	0.05	-0.11	-0.03	-0.15	-0.19	-0.16	-0.38
CreditBB	-0.57	-0.63	-0.35	-0.23	-0.51	-0.45	-0.31	-0.43	-0.07	0.25	0.29	0.34	0.47	0.64	1.00	0.37	0.00	0.00	-0.26	-0.05	-0.04	0.04	0.08	0.21	-0.08	-0.15	0.02	-0.17	-0.04	0.20	-0.24
CreditBB	-0.10	-0.16	0.05	0.05	-0.18	-0.68	-0.85	-0.49	-0.25	0.48	0.48	0.58	0.56	0.65	0.65	1.00	0.37	0.00	0.14	-0.06	-0.13	0.26	0.00	-0.02	-0.11	-0.13	0.29	0.20	-0.23	-0.39	-0.37
CreditEM	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
CurrEUR	-0.04	-0.08	-0.23	-0.27	-0.16	-0.04	0.05	-0.08	0.15	0.01	-0.15	-0.15	-0.18	-0.13	-0.26	-0.14	0.00	1.00	0.28	-0.20	0.08	0.12	0.32	0.22	0.46	0.39	0.34	0.11	0.03	0.00	
CurrGBP	-0.37	-0.31	-0.14	-0.47	-0.44	-0.16	0.18	-0.19	0.00	-0.29	-0.19	-0.12	-0.15	-0.08	-0.05	-0.06	0.00	0.28	1.00	0.01	0.31	0.43	0.41	0.07	0.20	0.07	0.00	-0.11	0.10	-0.21	
CurrCHF	0.08	0.12	0.09	0.13	0.17	0.14	0.04	0.25	0.07	-0.05	-0.03	-0.08	0.00	-0.12	-0.04	-0.13	0.00	-0.20	0.01	1.00	0.15	0.07	-0.35	-0.08	-0.05	-0.11	-0.04	-0.12	-0.23	0.02	
CurrSEK	-0.31	-0.43	-0.18	-0.38	-0.31	0.00	-0.02	-0.15	-0.14	-0.03	0.11	0.11	0.14	0.18	0.04	0.26	0.00	0.08	0.31	0.15	1.00	0.25	-0.06	0.08	0.04	-0.17	-0.06	-0.14	-0.05	-0.11	
CurrDKK	-0.13	-0.08	-0.12	-0.46	-0.03	0.03	0.07	-0.19	0.13	0.10	0.07	-0.05	0.16	0.25	0.08	0.00	0.00	0.12	0.43	0.07	0.25	1.00	0.14	0.12	0.03	-0.03	-0.11	-0.01	0.06	0.02	
CurrNOK	-0.08	-0.06	0.12	-0.41	-0.44	-0.11	0.05	0.11	0.32	0.30	0.21	0.14	0.21	0.26	0.21	-0.02	0.00	0.32	0.41	-0.35	-0.06	0.14	1.00	0.56	0.20	0.51	0.34	-0.32	0.35	0.07	
CurrUSD	0.01	0.04	0.02	-0.18	-0.07	-0.04	0.09	0.08	0.19	0.03	0.06	0.03	0.01	0.05	-0.08	-0.11	0.00	0.22	0.07	-0.08	0.08	0.12	0.56	1.00	0.43	0.57	0.36	0.29	0.20	0.12	
CurrCAD	0.11	0.06	-0.11	0.03	-0.13	0.04	0.10	0.07	0.38	-0.24	0.00	-0.31	-0.25	-0.11	-0.15	-0.13	0.00	0.46	0.20	-0.05	0.04	0.03	0.20	0.43	1.00	0.50	0.52	0.33	0.10	0.25	
CurrJPY	0.28	0.16	-0.04	0.43	-0.06	0.03	0.16	-0.35	0.40	-0.02	0.00	-0.13	-0.10	-0.03	0.02	0.29	0.00	0.39	0.07	-0.11	-0.17	-0.03	0.51	0.57	0.50	1.00	0.91	0.42	0.29	0.39	
CurrAUD	-0.06	-0.11	-0.09	0.07	-0.10	0.07	0.09	0.00	0.26	-0.18	-0.22	-0.18	-0.23	-0.15	-0.17	0.20	0.00	0.34	0.00	-0.04	-0.06	-0.11	0.34	0.36	0.52	0.91	1.00	0.26	0.14	0.13	
CurrNZD	0.03	0.02	-0.22	-0.19	-0.21	0.10	0.22	0.05	0.13	-0.21	-0.31	-0.34	-0.23	-0.19	-0.04	-0.23	0.00	0.11	-0.11	-0.12	-0.14	-0.01	0.32	0.29	0.33	0.42	0.26	1.00	0.86	0.79	
CurrSGD	-0.31	-0.30	-0.37	-0.26	-0.45	0.12	0.37	0.04	-0.04	-0.27	-0.31	-0.29	-0.22	-0.16	0.20	-0.39	0.00	0.03	0.10	-0.23	-0.05	0.06	0.35	0.20	0.10	0.29	0.14	0.86	1.00	0.74	
CurrHKD	0.06	0.11	-0.19	0.13	0.15	0.18	0.27	0.15	0.07	-0.33	-0.42	-0.42	-0.38	-0.38	-0.24	-0.37	0.00	0.00	-0.21	0.02	-0.11	0.02	0.07	0.12	0.25	0.39	0.13	0.79	0.74	1.00	
np_Vola	1.13%	0.95%	0.90%	1.29%	1.31%	3.2701	4.4622	3.2131	0.5712	3.2187	1.2531	1.2356	1.3208	1.4993	3.7716	2.9383	0.00%	0.21%	0.14%	0.22%	0.01%	0.40%	0.54%	0.51%	0.38%	0.34%	0.79%	0.36%	0.50%	0.53%	
Exposures	10,000	0	60,000	30,000	0	-100	-550	-250	-200	-400	-800	400	300	900	-200	-900	910,000	0	0	0	0	0	0	0	60,000	0	30,000	0	0	0	0
Equ EUR	1.00	0.98	0.62	0.52	0.77	0.14	0.18	0.39	0.48	0.19	-0.10	-0.21	-0.02	-0.05	-0.57	-0.10	0.00	0.00	-0.04	-0.37	0.08	-0.31	-0.13	-0.08	0.01	0.11	0.28	-0.06	0.03	-0.31	0.06
Equ exEUR	0.98	1.00	0.67	0.48	0.82	0.15	-0.17	0.44	0.53	0.21	-0.06	-0.17	-0.05	-0.13	-0.63	-0.16	0.00	0.00	-0.08	-0.31	0.12	-0.43	-0.08	-0.06	0.04	0.06	0.16	-0.11	0.02	-0.30	0.11
Equ NA	0.62	0.67	1.00	0.16	0.51	0.02	-0.09	0.29	0.00	0.22	0.18	0.18	0.09	0.00	-0.35	0.05	0.00	0.00	-0.23	-0.14	0.09	-0.18	-0.12	0.12	0.02	-0.11	-0.04	-0.09	-0.22	-0.37	-0.19
Equ AsP	0.52	0.48	0.16	1.00	0.78	0.09	-0.19	0.26	0.44	0.05	-0.09	-0.07	-0.20	-0.22	-0.23	0.05	0.00	0.00	0.27	0.47	0.13	-0.38	-0.46	-0.41	-0.18	0.03	0.43	0.07	-0.19	-0.26	0.13
Equ EM	0.77	0.82	0.51	0.78	1.00	0.18	-0.16	0.33	0.29	0.08	0.06	0.01	-0.10	-0.20	-0.51	-0.18	0.00	0.00	0.16	0.44	0.17	-0.31	-0.03	-0.44	-0.07	-0					

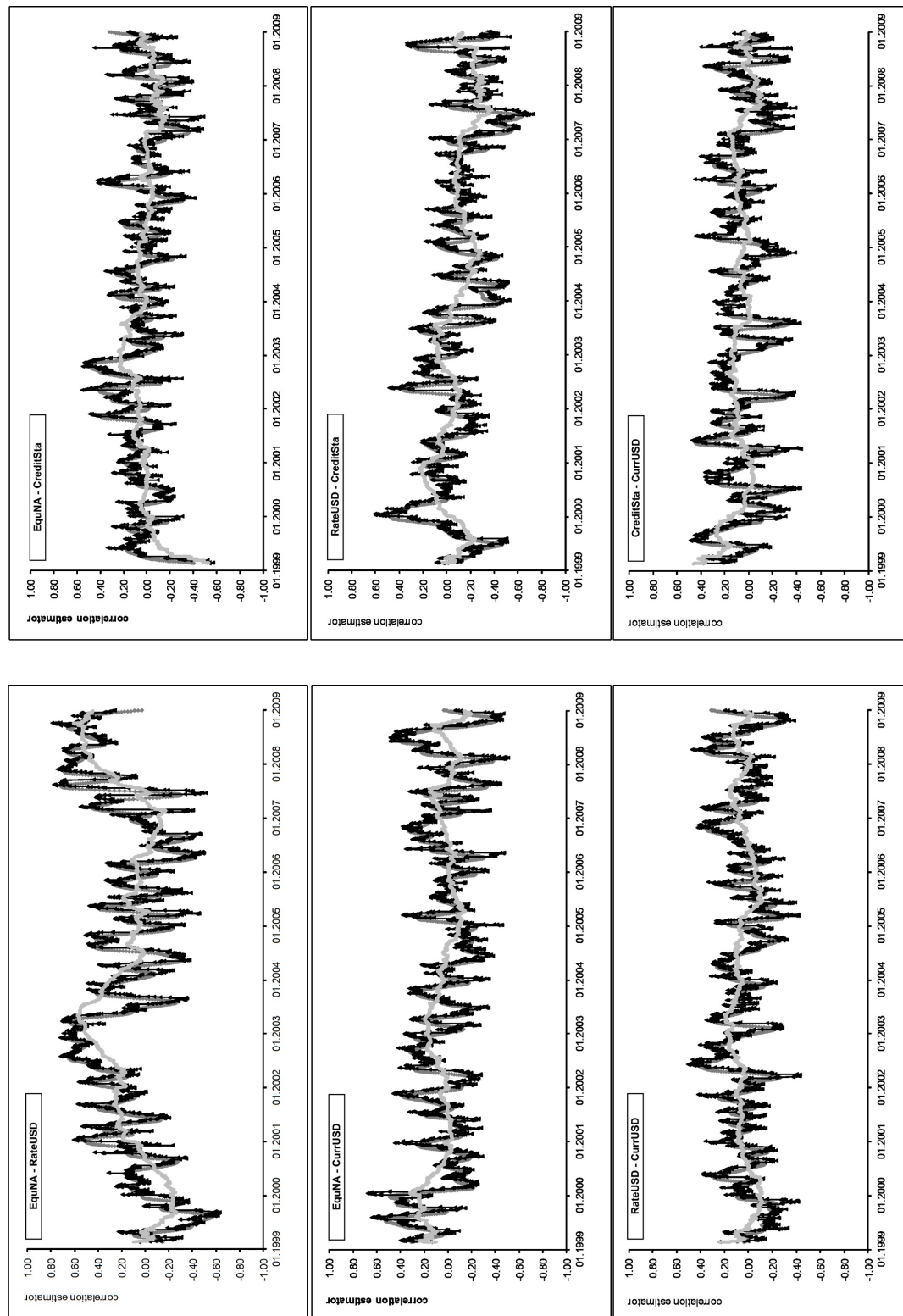


Figure 12: Correlation estimates regarding daily returns of the pairs of series *EquiNA*, *RateUSD*, *CredSta* and *CurrUSD*. From the wholistic smoothing of regression model (2) the two-sided standardized MNWEs (based on bandwidth  $\bar{h}^{\text{opt}} = 33$  days) are depicted in dark gray bullets and the one-sided standardized MNWEs (based on  $\bar{h}_{(1)}^{\text{opt}} = 47$  days) in black triangles. The 258-day empirical correlation (bright gray line) is opposed each with.

timators. To comment some examples, the *EquNA-RateUSD* correlation is about  $-0.3$  in November 1999 regarding both nonparametric estimates, and exhibits a positive trend the following two years (noticeable also in the 258-day empirical correlation). A plateau of around  $0.6$  results from middle 2002, followed by a significant decline in spring 2003. After volatile years a significant positive interdependence is achieved again from summer 2007. In contrast to individual smoothing, here the two-sided and the one-sided graphs almost cover each other and have very similar dynamics and peaks. This feature is confirmed by the other examples, speaking in favor for an adequate relation between two- and one-sided bandwidths. The heteroscedasticity of the *EquNA-RateUSD* correlation can be proved again by record dates November 15, 1999 ( $\hat{\rho} = 0.40, \hat{\rho}_{(1)} = 0.45$ ) versus May 15, 2003 ( $\hat{\rho} = -0.17, \hat{\rho}_{(1)} = -0.19$ ); empirical correlations lag behind on both dates ( $0.24$  the former,  $0.15$  the latter date). Although the nonparametric time-dependence of *RateUSD-CurrUSD* is a bit more jagged than before, phases of weakly negative correlations ( $\hat{\rho} = -0.17, \hat{\rho}_{(1)} = -0.35$  on November 15, 1999) and weakly positive correlations ( $\hat{\rho} = 0.10, \hat{\rho}_{(1)} = 0.27$  on May 15, 2003) can be distinguished.

Continuing on the representative two dates, figures 13 and 14 document the variation of the  $30 \times 30$  correlation matrices over time for the two-sided and the one-sided nonparametric estimates  $\hat{\boldsymbol{\rho}}_{(\cdot)h}(t)$  (on lower triangular) against empirical estimators (on upper triangular). The respective correlation differences between November 15, 1999 and May 15, 2003 are calculated in the bottom tableau of each figure, where deviations being significant to a (conservative) 5% level are highlighted in gray. Regarding the two-sided nonparametric estimators about 19% of the realizations differ significantly, where its average absolute deviation is  $0.20$ . About 14% of the time differences of one-sided estimates are statistically significant (av. abs. dev.  $0.21$ ). Compared to the statistics provided for individual smoothing (figures 9 and 10) the fraction of significant deviations and the average absolute deviation of correlations are reduced due to wholistic bandwidths. The one-sided approach varies still more than its two-sided counterpart or than the empirical correlation (av. abs. dev.  $0.13$ , 25% significantly time-dependent), but in combination with an increased smoothness there are less significances. Overall the examples endorse the effort for modeling time-variant correlations with our non-stationary regression model.

## 4 Empirical Studies

Having motivated the need of the wholistic smoothing approach and gained nonparametric estimates of covariance matrices in the previous chapter, we finish the non-stationary modeling (2) of financial times series regarding the 10-year daily returns in the first part of this section. In the second part we apply the regression approach to forecast multivariate return distributions in exemplary portfolios with 30 exposure classes, and we evaluate the model fit in terms of a backtesting and a comparison to standard risk models.

### 4.1 Complete model approximation for financial time series

Using the wholistic nonparametric covariance estimates  $\hat{\boldsymbol{\Sigma}}^2(t)$  and  $\hat{\boldsymbol{\Sigma}}_{(1)}^2(t)$  for the 30 daily benchmark series from January 1999 to December 2008, obtained at the end of section 3.2, the remaining task is to fit the multivariate return distribution in its return residuals. By dint of demeaned returns and nonparametric

**15.11.99** empirical estimates

Corr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Swa	Credit AAA	Credit AA	Credit A	Credit BBB	Credit BB	Credit B	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM
EquEUR	1.00	0.86	0.45	0.33	0.50	-0.18	-0.09	0.03	0.17	0.09	0.10	0.14	0.12	0.12	-0.42	0.07	0	0.22	-0.26	0.27	0.04	0.31	0.40	0.41	0.08	0.32	0.21	0.30	0.28	0.36
EquexEUR	0.82	1.00	0.41	0.34	0.48	-0.10	-0.04	-0.01	0.13	0.00	0.01	0.05	0.01	0.03	-0.38	0.02	0	0.21	-0.17	0.24	0.01	0.22	0.32	0.31	0.01	0.24	0.14	0.26	0.24	0.33
EquNA	0.54	0.53	1.00	0.08	0.32	-0.22	-0.24	0.00	-0.01	0.08	0.04	0.07	0.07	0.09	-0.31	0.21	0	0.15	-0.06	0.08	0.07	0.10	0.24	0.13	0.06	0.13	0.06	0.32	0.39	0.43
EquASP	0.28	0.24	0.32	1.00	0.45	-0.01	-0.03	0.01	0.17	0.02	0.02	0.04	0.06	0.06	-0.08	0.00	0	0.10	-0.23	0.17	0.03	0.19	0.18	0.26	0.02	0.12	0.12	0.00	-0.06	-0.01
EquEM	0.44	0.52	0.43	0.40	1.00	0.03	-0.02	0.01	0.07	-0.13	-0.09	-0.02	-0.05	-0.07	-0.52	-0.06	0	0.12	-0.40	0.26	0.06	0.36	0.25	0.33	0.10	0.26	0.22	0.22	0.33	
RateEUR	-0.08	-0.08	-0.29	0.12	-0.21	1.00	0.38	0.11	0.21	0.01	-0.62	-0.41	-0.34	-0.38	-0.32	0.03	-0.40	0	0.00	-0.04	-0.01	0.04	0.07	-0.01	0.00	0.06	0.06	-0.09	-0.07	-0.11
RateUSD	-0.17	-0.16	-0.26	-0.03	-0.20	0.61	1.00	0.09	-0.13	-0.22	-0.09	-0.12	-0.12	-0.06	-0.15	-0.82	0	-0.10	-0.07	-0.05	-0.08	0.02	-0.11	-0.06	-0.11	0.01	-0.05	-0.01	-0.03	0.00
RateJPY	0.03	0.00	0.07	0.34	-0.10	0.35	0.13	1.00	0.09	0.10	-0.03	-0.10	-0.07	-0.09	-0.06	-0.04	0	-0.02	0.01	0.06	0.11	0.03	0.02	0.01	0.02	0.10	-0.03	-0.03	0.02	
CreditSta	0.04	0.01	-0.11	0.21	0.06	0.42	0.16	0.06	1.00	0.27	-0.10	-0.13	-0.13	-0.22	-0.07	0.15	0	0.20	-0.09	0.14	0.03	0.13	0.24	0.27	0.13	0.16	0.16	0.01	0.02	0.03
CreditSwa	0.00	-0.04	0.20	-0.08	-0.01	-0.62	-0.23	-0.10	-0.77	1.00	0.50	0.43	0.47	0.35	0.03	0.28	0	0.06	0.10	0.03	-0.06	0.02	0.07	0.01	0.02	0.00	-0.02	0.01	0.02	0.03
CreditAAA	-0.16	-0.15	-0.07	-0.08	-0.14	-0.41	-0.10	0.00	-0.66	0.59	1.00	0.90	0.89	0.79	0.03	0.12	0	0.01	0.06	0.08	-0.06	0.04	0.00	0.01	-0.04	0.03	-0.01	0.00	-0.01	-0.03
CreditAA	-0.05	0.03	-0.03	-0.16	-0.05	-0.43	-0.19	-0.11	-0.68	0.55	0.89	1.00	0.87	0.73	0.02	0.14	0	0.00	0.02	0.12	-0.04	0.05	0.02	0.02	-0.06	0.05	0.00	0.02	0.01	0.01
CreditA	-0.15	-0.11	-0.05	-0.11	-0.18	-0.32	-0.03	-0.09	-0.62	0.50	0.84	0.83	1.00	0.74	0.00	0.13	0	0.02	0.04	0.06	-0.05	0.03	0.02	-0.01	-0.04	0.02	0.00	0.02	0.02	0.01
CreditBBB	0.01	0.03	0.09	-0.11	-0.06	-0.28	-0.10	-0.14	-0.46	0.34	0.49	0.44	1.00	1.00	0.01	0.08	0	0.02	0.05	0.01	-0.06	-0.01	-0.02	-0.04	-0.10	0.00	0.00	0.00	-0.01	-0.02
CreditBB	-0.44	-0.47	-0.45	-0.24	-0.40	0.10	0.00	-0.13	-0.22	-0.12	-0.03	-0.04	-0.09	0.06	1.00	0.15	0	-0.11	0.29	-0.15	-0.07	-0.31	-0.18	-0.21	-0.16	-0.19	-0.19	-0.24	-0.22	-0.41
CreditB	0.11	0.07	0.21	0.04	0.04	-0.58	-0.80	-0.15	-0.19	0.29	0.21	0.24	0.12	0.12	0.04	0.00	0	0.08	0.10	-0.04	0.02	-0.11	0.11	0.00	0.03	-0.04	-0.04	-0.01	0.01	-0.03
CurrEUR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
CurrGBP	0.33	0.26	0.28	0.17	0.26	0.01	-0.06	-0.12	0.04	0.01	-0.24	-0.16	-0.22	-0.04	-0.24	0.06	0	1.00	-0.07	0.28	0.14	0.35	0.67	0.55	0.38	0.51	0.49	0.27	0.31	0.25
CurrCHF	-0.22	-0.26	-0.23	-0.23	0.37	0.11	0.11	0.02	-0.18	0.10	0.10	0.08	0.09	0.21	0.25	0.03	0	-0.10	1.00	-0.17	-0.02	-0.20	-0.17	-0.29	-0.08	-0.17	-0.19	-0.13	-0.08	-0.09
CurrSEK	0.22	0.09	0.09	0.10	0.22	0.03	-0.24	0.15	0.11	-0.02	-0.19	-0.12	-0.05	-0.19	-0.16	0.13	0	0.29	-0.20	1.00	0.21	0.65	0.42	0.43	0.24	0.41	0.33	0.11	0.15	0.14
CurrDKK	-0.09	-0.06	0.08	0.08	0.11	0.01	0.03	-0.10	0.07	-0.01	-0.05	-0.07	-0.07	-0.02	-0.07	-0.09	0	0.21	-0.07	0.13	1.00	0.14	0.15	0.17	0.15	0.16	0.14	0.05	0.10	0.11
CurrNOK	0.45	0.34	0.21	0.27	0.33	0.18	0.07	0.28	0.12	-0.17	-0.29	-0.27	-0.30	-0.22	-0.35	0.01	0	0.43	-0.19	0.61	0.11	1.00	0.52	0.55	0.32	0.50	0.45	0.18	0.23	0.24
CurrUSD	0.26	0.22	0.40	0.25	0.26	-0.06	-0.17	0.03	-0.02	0.07	-0.19	-0.14	-0.24	-0.11	-0.22	0.21	0	0.74	-0.18	0.44	0.18	0.52	1.00	0.83	0.44	0.66	0.57	0.28	0.34	0.29
CurrCAD	0.27	0.22	0.34	0.34	0.29	0.07	-0.28	0.10	-0.02	0.08	-0.11	-0.07	-0.20	-0.11	-0.27	0.26	0	0.66	-0.34	0.47	0.10	0.53	0.90	1.00	0.35	0.74	0.65	0.32	0.35	0.34
CurrJPY	0.05	0.10	0.08	0.03	0.08	0.13	0.06	0.02	0.06	-0.08	-0.20	-0.14	-0.18	-0.23	-0.11	0.04	0	0.58	-0.12	0.33	0.17	0.48	0.67	0.58	1.00	0.35	0.36	0.22	0.14	0.19
CurrAUD	0.26	0.18	0.19	0.20	0.18	0.05	0.04	0.06	0.09	-0.04	-0.37	-0.26	-0.31	-0.08	-0.18	0.03	0	0.64	-0.15	0.48	0.21	0.47	0.75	0.72	0.46	1.00	0.82	0.21	0.24	0.25
CurrNZD	0.24	0.23	0.12	0.17	0.16	0.06	-0.12	0.00	-0.15	-0.11	-0.30	-0.18	-0.22	-0.10	-0.10	0.10	0	0.45	-0.20	0.35	0.17	0.40	0.58	0.60	0.41	0.75	1.00	0.12	0.15	0.14
CurrSGD	0.18	0.29	0.38	0.07	0.17	-0.14	-0.04	0.00	-0.19	0.18	0.00	-0.03	-0.05	-0.02	-0.44	0.09	0	0.37	-0.19	0.06	0.00	0.23	0.42	0.41	0.32	0.13	0.14	1.00	0.84	0.82
CurrHKD	0.17	0.30	0.36	0.01	0.23	-0.07	0.08	-0.02	-0.16	0.14	-0.04	-0.05	-0.05	-0.08	-0.42	-0.04	0	0.44	-0.24	0.12	0.08	0.29	0.52	0.49	0.43	0.25	0.24	0.91	1.00	0.89
CurrEM	0.24	0.35	0.44	0.06	0.28	-0.10	0.03	-0.04	-0.17	0.15	0.01	-0.01	-0.01	-0.05	-0.48	0.00	0	0.43	-0.26	0.09	0.07	0.28	0.47	0.45	0.36	0.18	0.20	0.92	0.97	1.00

two-sided nonparametric estimates

**15.05.03** empirical estimates

Corr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Swa	Credit AAA	Credit AA	Credit A	Credit BBB	Credit BB	Credit B	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY	Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM
EquEUR	1.00	0.93	0.61	0.23	0.45	0.66	0.46	0.10	0.26	-0.21	-0.12	-0.29	-0.39	-0.46	-0.45	-0.50	0	0.13	-0.24	0.36	0.03	0.16	0.31	0.37	0.00	0.27	0.20	0.25	0.30	0.35
EquexEUR	0.90	1.00	0.51	0.26	0.47	0.56	0.38	0.11	0.26	-0.11	-0.06	-0.23	-0.31	-0.43	-0.36	-0.41	0	0.12	-0.24	0.41	0.03	0.16	0.33	0.40	-0.01	0.30	0.22	0.19	0.25	0.29
EquNA	0.68	0.69	1.00	0.11	0.29	0.49	0.56	0.13	0.14	-0.26	-0.11	-0.18	-0.27	-0.21	-0.44	-0.56	0	0.05	-0.07	0.14	0.03	0.07	0.15	0.23	0.04	0.11	0.05	0.36	0.40	0.39
EquASP	0.26	0.33	0.07	1.00	0.80	0.13	0.06	0.05	0.16	0.24	0.08	-0.07	-0.17	-0.35	-0.16	-0.17	0	0.03	-0.23	0.24	-0.08	0.07	0.18	0.22	0.01	0.27	0.23	-0.01	-0.02	0.03
EquEM	0.47	0.49	0.34	0.52	1.00	0.27	0.23	0.09	0.22	0.19	0.16	-0.07	-0.23	-0.44	-0.38	-0.35	0	0.13	-0.38	0.25	-0.03	0.10	0.32	0.36	0.10	0.35	0.30	0.09	0.07	0.15
RateEUR	0.48	0.40	0.39	0.13	0.22	1.00	0.57	0.11	0.28	-0.60	-0.35	-0.46	-0.52	-0.46	-0.33	-0.54	0	0.15	-0.13	0.24	0.04	0.09	0.27	0.30	0.06	0.17	0.17	0.34	0.40	0.39
RateUSD	-0.01	-0.04	0.15	-0.19	0.13	0.43	1.00	0.14	0.06	-0.29	-0.15	-0.20	-0.28	-0.24	-0.46	-0.64	0	0.09	-0.16	0.11	0.10	0.10	0.19	0.18	0.04	0.07	0.06	0.32	0.35	0.34
RateJPY	-0.11	-0.09	0.07	-0.04	-0.07	0.20	0.29	1.00	0.03	-0.04	0.06	0.00	-0.11	-0.15	-0.08	-0.15	0	0.08	-0.03	0.04	-0.04	-0.10	0.17	0.16	0.06	-0.09	0.05	0.12	0.18	0.12
CreditSta	-0.06	-0.09	-0.09	0.01	0.11	0.12	0.09	0.10	1.00	-0.08	0.03	-0.06	-0.16	-0.25	-0.08	-0.18	0	0.03	-0.04	0.25	0.07	0.07	0.12	0.19	0.04	0.25	0.23	0.07	0.04	0.09
CreditSwa	-0.19	-0.04	-0.39	0.34	-0.03	-0.57	-0.36	-0.01	-0.16	1.00	0.68	0.55	0.51	0.25	0.13	0.25	0	0.18	-0.13	0.05	-0.04	0.03	0.18	0.10	0.02	0.12	0.08	-0.26	-0.29	-0.26
CreditAAA	-0.03	0.13	-0.16	0.27	0.08	-0.36	-0.28	-0.09	-0.04	0.74	1.00	0.85	0.71	0.35	0.07	0.11	0	0.26	-0.14	0.06	-0.05	0.07	0.26	0.20	0.09	0.19	0.14	-0.07	-0.09	-0.06
CreditAA	-0.26	-0.13	-0.27	0.10	-0.12	-0.52	-0.31	-0.09	-0.07	0.69	0.86	1.00	0.85	0.51	0.19	0.23	0	0.13	-0.04	-0.09	-0.02	0.05	0.08	-0.01	0.08	0.09	0.05	-0.04	-0.09	-0.

15.11.99 empirical estimates																										empirical estimates					
Curr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Swa	Credit AAA	Credit AA	Credit A	Credit BBB	Credit BB	Credit B	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY		Curr AUD	Curr NZD	Curr SGD	Curr HKD	Curr EM
EquEUR	1.00	0.86	0.45	0.33	0.50	-0.18	-0.09	0.03	0.17	0.09	0.10	0.14	0.12	0.12	-0.42	0.07	0	0.22	-0.26	0.27	0.04	0.31	0.40	0.41	0.08	0.32	0.21	0.30	0.28	0.36	
EquexEUR	0.92	1.00	0.41	0.34	0.48	-0.10	-0.04	-0.01	0.13	0.00	0.01	0.05	0.01	0.03	-0.38	0.02	0	0.21	-0.17	0.24	0.01	0.22	0.32	0.31	0.01	0.24	0.14	0.26	0.24	0.33	
EquNA	0.59	0.55	1.00	0.08	0.32	-0.22	-0.24	0.00	-0.01	0.08	0.04	0.07	0.07	0.09	-0.31	0.21	0	0.15	-0.06	0.08	0.07	0.10	0.24	0.13	0.06	0.13	0.06	0.32	0.39	0.43	
EquASP	0.33	0.32	0.31	1.00	0.45	-0.01	-0.03	0.01	0.17	0.02	0.02	0.04	0.06	0.06	-0.08	0.00	0	0.10	-0.23	0.17	0.03	0.19	0.18	0.26	0.02	0.12	0.12	0.00	-0.06	-0.01	
EquEM	0.53	0.60	0.43	0.52	1.00	0.03	-0.02	0.01	0.07	-0.13	-0.09	-0.02	-0.05	-0.07	-0.52	-0.06	0	0.12	-0.40	0.26	0.06	0.36	0.25	0.33	0.10	0.26	0.22	0.22	0.23	0.33	
RateEUR	-0.23	-0.13	-0.41	0.06	-0.01	1.00	0.38	0.11	0.21	-0.62	-0.41	-0.34	-0.38	-0.32	0.03	-0.40	0	0.00	-0.04	-0.01	0.04	0.07	-0.01	0.00	0.06	0.06	0.06	-0.09	-0.07	-0.11	
RateUSD	-0.33	-0.29	-0.36	-0.13	-0.27	0.53	1.00	0.09	-0.13	-0.22	-0.09	-0.12	-0.12	-0.06	-0.15	-0.82	0	-0.10	-0.07	-0.05	-0.08	0.02	-0.11	-0.06	-0.11	0.01	-0.05	-0.01	-0.03	0.00	
RateJPY	0.09	0.05	0.04	0.34	0.28	0.20	0.10	1.00	0.09	0.10	-0.03	-0.10	-0.07	-0.09	-0.06	-0.04	0	-0.02	0.01	0.06	0.11	0.03	0.02	0.01	0.02	0.00	0.02	0.10	-0.03	0.03	0.02
CreditSta	0.12	0.10	0.07	0.21	0.24	0.18	-0.02	0.21	1.00	0.27	-0.10	-0.13	-0.13	-0.22	-0.07	0.15	0	0.20	-0.09	0.14	0.03	0.13	0.24	0.27	0.13	0.16	0.16	0.01	0.02	0.03	
CreditSwa	0.01	-0.08	0.13	0.03	-0.20	-0.68	-0.20	-0.06	-0.35	1.00	0.50	0.43	0.47	0.35	0.03	0.28	0	0.06	0.10	0.03	-0.06	0.02	0.07	0.01	0.02	0.00	-0.02	0.01	0.02	0.03	
CreditAAA	-0.13	-0.15	-0.15	-0.02	-0.22	-0.33	-0.13	-0.04	-0.60	0.46	1.00	0.90	0.89	0.79	0.03	0.12	0	0.01	0.06	0.08	-0.06	0.04	0.00	0.01	-0.04	0.03	-0.01	0.00	-0.01	-0.03	
CreditAA	-0.01	0.01	-0.09	-0.03	-0.09	-0.27	-0.23	0.05	-0.62	0.34	0.91	1.00	0.87	0.73	0.02	0.14	0	0.00	0.02	0.12	-0.04	0.05	0.02	0.02	-0.06	0.05	0.00	0.02	0.01	0.01	
CreditA	-0.01	-0.10	-0.05	0.02	-0.21	-0.23	-0.04	0.06	-0.60	0.42	0.86	0.81	1.00	0.74	0.00	0.13	0	0.02	0.04	0.06	-0.05	0.03	0.02	-0.01	-0.04	0.02	0.00	0.02	0.02	0.01	
CreditBBB	0.02	-0.07	-0.08	-0.01	-0.27	-0.26	-0.17	-0.08	-0.61	0.52	0.79	0.75	0.77	1.00	0.01	0.08	0	0.02	0.05	0.01	-0.06	-0.01	-0.02	-0.04	-0.10	0.05	0.00	0.00	0.01	-0.02	
CreditBB	-0.64	-0.64	-0.40	-0.19	-0.40	0.16	0.02	-0.02	0.01	0.01	-0.01	-0.08	-0.16	-0.03	1.00	0.15	0	-0.11	0.29	-0.15	-0.07	-0.31	-0.18	-0.21	-0.16	-0.19	-0.19	-0.24	-0.22	-0.41	
CreditB	0.32	0.24	0.38	0.18	0.04	-0.52	-0.80	-0.16	0.04	0.23	0.18	0.21	0.17	0.25	-0.08	1.00	0	0.08	0.10	-0.04	0.02	-0.11	0.11	0.00	0.03	-0.04	-0.04	-0.01	0.01	-0.03	
CurrEUR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	
CurrGBP	0.24	0.21	0.23	0.11	0.02	0.01	-0.11	-0.16	0.12	-0.12	-0.19	-0.20	-0.12	-0.04	-0.20	0.20	0	1.00	-0.07	0.28	0.14	0.35	0.67	0.55	0.38	0.51	0.49	0.27	0.31	0.25	
CurrCHF	-0.37	-0.40	-0.37	-0.30	-0.39	0.07	0.12	-0.14	-0.15	0.17	0.11	-0.01	0.06	0.27	0.35	-0.08	0	-0.23	1.00	-0.17	-0.02	-0.20	-0.17	-0.29	-0.08	-0.17	-0.19	-0.13	-0.08	-0.09	
CurrSEK	0.29	0.23	0.17	0.17	0.24	0.05	-0.14	0.28	0.27	-0.16	-0.20	-0.08	-0.06	-0.10	-0.09	0.06	0	0.45	-0.27	1.00	0.21	0.65	0.42	0.43	0.24	0.41	0.33	0.11	0.15	0.14	
CurrDKK	-0.18	-0.09	-0.10	-0.02	-0.05	0.12	0.07	0.10	0.03	-0.15	-0.03	-0.05	-0.16	-0.09	-0.09	-0.05	0	0.17	0.12	0.22	1.00	0.14	0.15	0.17	0.15	0.16	0.14	0.05	0.10	0.11	
CurrNOK	0.29	0.26	0.16	0.26	0.35	0.23	0.04	0.19	0.33	-0.38	-0.36	-0.35	-0.29	-0.30	-0.16	-0.02	0	0.35	-0.24	0.62	0.09	1.00	0.52	0.55	0.32	0.50	0.45	0.18	0.23	0.24	
CurrUSD	0.37	0.32	0.45	0.30	0.22	-0.18	-0.35	-0.03	0.11	-0.09	-0.14	-0.08	-0.09	-0.03	-0.23	0.45	0	0.70	-0.38	0.58	0.16	0.44	1.00	0.83	0.44	0.66	0.57	0.28	0.34	0.29	
CurrCAD	0.38	0.34	0.35	0.37	0.32	-0.17	-0.31	0.03	0.10	-0.07	-0.07	-0.02	-0.06	-0.06	-0.30	0.34	0	0.57	-0.58	0.63	0.10	0.46	0.87	1.00	0.35	0.74	0.65	0.32	0.35	0.34	
CurrJPY	0.07	0.06	0.12	-0.01	-0.03	0.07	-0.19	0.02	0.19	-0.25	-0.18	-0.15	-0.18	-0.18	0.04	0.22	0	0.65	-0.26	0.38	0.19	0.31	0.59	0.48	1.00	0.35	0.36	0.22	0.14	0.19	
CurrAUD	0.30	0.23	0.19	0.15	0.15	0.05	0.05	0.06	0.12	-0.17	-0.33	-0.21	-0.21	-0.14	-0.27	0.11	0	0.56	-0.17	0.64	0.23	0.43	0.70	0.68	0.30	1.00	0.82	0.21	0.24	0.25	
CurrNZD	0.44	0.30	0.23	0.34	0.29	-0.01	-0.32	0.08	0.23	-0.16	-0.23	-0.15	-0.20	-0.07	-0.25	0.29	0	0.45	-0.19	0.51	0.10	0.40	0.63	0.68	0.28	0.78	1.00	0.12	0.15	0.14	
CurrSGD	0.29	0.35	0.47	0.04	0.11	-0.29	-0.01	-0.17	-0.12	0.10	0.01	-0.01	0.05	0.00	-0.47	0.10	0	0.25	-0.32	0.01	-0.05	0.06	0.18	0.21	0.21	-0.02	-0.04	1.00	0.84	0.82	
CurrHKD	0.25	0.31	0.47	0.04	0.20	-0.28	0.01	-0.24	-0.05	0.08	-0.05	-0.08	-0.01	-0.11	-0.48	0.09	0	0.31	-0.39	0.11	-0.01	0.18	0.29	0.35	0.25	0.08	0.00	0.87	1.00	0.89	
CurrEM	0.31	0.36	0.54	0.06	0.27	-0.31	-0.02	-0.17	-0.10	0.08	-0.01	-0.04	0.04	-0.08	-0.52	0.11	0	0.29	-0.45	0.09	-0.03	0.18	0.29	0.35	0.23	0.08	0.02	0.90	0.97	1.00	

15.05.03 empirical estimates																										empirical estimates				
Curr	Equ EUR	Equ exEUR	Equ NA	Equ ASP	Equ EM	Rate EUR	Rate USD	Rate JPY	Credit Sta	Credit Swa	Credit AAA	Credit AA	Credit A	Credit BBB	Credit BB	Credit B	Curr EUR	Curr GBP	Curr CHF	Curr SEK	Curr DKK	Curr NOK	Curr USD	Curr CAD	Curr JPY		Curr AUD	Curr NZD	Curr SGD	Curr HKD
EquEUR	1.00	0.93	0.61	0.23	0.45	0.66	0.46	0.10	0.26	-0.21	-0.12	-0.29	-0.39	-0.46	-0.45	-0.50	0	0.13	-0.24	0.36	0.03	0.16	0.31	0.37	0.00	0.27	0.20	0.25	0.30	0.35
EquexEUR	0.93	1.00	0.51	0.26	0.47	0.56	0.38	0.11	0.26	-0.11	-0.06	-0.23	-0.31	-0.43	-0.36	-0.41	0	0.12	-0.24	0.41	0.03	0.16	0.33	0.40	-0.01	0.30	0.22	0.19	0.25	0.29
EquNA	0.68	0.70	1.00	0.11	0.29	0.49	0.56	0.13	0.14	-0.26	-0.11	-0.18	-0.27	-0.21	-0.44	-0.56	0	0.05	-0.07	0.14	0.03	0.07	0.15	0.23	0.04	0.11	0.05	0.36	0.40	0.39
EquASP	0.24	0.30	0.08	1.00	0.80	0.13	0.06	0.05	0.16	0.24	0.08	-0.07	-0.17	-0.35	-0.16	-0.17	0	0.03	-0.23	0.24	-0.08	0.07	0.18	0.22	0.01	0.27	0.23	-0.01	-0.02	0.03
EquEM	0.52	0.53	0.35	0.47	1.00	0.27	0.23	0.09	0.22	0.19	0.16	-0.07	-0.23	-0.44	-0.38	-0.35	0	0.13	-0.38	0.25	-0.03	0.10	0.32	0.36	0.10	0.35	0.30	0.09	0.07	0.15
RateEUR	0.51	0.45	0.44	0.01	0.32	1.00	0.57	0.11	0.28	-0.60	-0.35	-0.46	-0.32	-0.46	-0.33	-0.54	0	0.15	-0.13	0.24	0.04	0.09	0.27	0.30	0.06	0.17	0.17	0.34	0.40	0.39
RateUSD	0.29	0.29	0.40	-0.14	0.16	0.63	1.00	0.14	0.06	-0.29	-0.15	-0.20	-0.28	-0.24	-0.46	-0.64	0	0.09	-0.16	0.11	0.10	0.19	0.18	0.04	0.07	0.06	0.32	0.35	0.34	
RateJPY	0.29	0.37	0.13	0.31	0.29	0.33	1.00	0.03	0.04	0.06	0.00	-0.11	-0.15	-0.08	-0.18	-0.15	0	0.08	-0.03	0.04	-0.04	-0.10	0.17	0.06	-0.09	0.05	0.05	0.12	0.18	0.12
CreditSta	-0.20	-0.24	-0.28	-0.09	-0.05	0.02	-0.08	-0.11	1.00	-0.08	0.03	-0.06	-0.16	-0.25	-0.08	-0.18	0	0.03	-0.04	0.25	0.07	0.07	0.12	0.19	0.04	0.25	0.23	0.07	0.04	0.09
CreditSwa	-0.16	-0.12	-0.35	0.32	0.05	-0.57	-0.47	0.11	-0.15	1.00	0.68	0.55	0.51	0.25	0.13	0.25	0	0.18	-0.13	0.05	-0.04	0.03	0.18	0.10	0.02	0.12	0.08	-0.26	-0.29	-0.26
CreditAAA	-0.21	-0.14	-0.14	0.01	-0.11	-0.31	-0.21	0.23	-0.13	0.55	1.00	0.85	0.71	0.35	0.07	0.11	0	0.26	-0.14	0.06	-0.05	0.07	0.26	0.20	0.09	0.19	0.14	-0.07	-0.09	-0.06
CreditAA	-0.37	-0.35	-0.25	-0.10	-0.27	-0.44	-0.25	0.06	-0.05	0.82	0.86	1.00	0.85	0.51	0.19	0.23	0	0.13	-0.04	-0.09	-0.02	0.05	0.08	-0.01	0.08	0.09				

volatility estimates the asymmetric Pearson VII density  $f_{m+,c+,m-,c-}^{VII}$  is fitted to the univariate series of realized innovations  $\hat{\varepsilon}_{k,1}, \dots, \hat{\varepsilon}_{k,n}$ , separately in each component  $k = 1, \dots, 30$ . The results are presented in figure 15 for the two-sided and the one-sided wholistic setup. Compared to individual smoothing there are a lot of more successes in fitting heavy-tailed residual components (failure rate only 15% vs. 44% afore). Regarding symmetrical information the complete Pearson VII fit is possible for 21 (of 29) benchmarks; in terms of the one-sided approach the complete model approximation is successful 22 times, where left tails are heavier with 28 fits. The tail index point estimate is lower 15 for the most right tails of interest rates and credit spreads, indicating an especially heavy loss distribution. For equities and currencies the asymmetries are less systematic and the tail index is bigger in general (being closer to Gaussian residuals).

Parameters for complete implementation of the multivariate non-stationary model										
<b>Two-sided setup</b>	<b>EquEUR</b>	<b>EquexEUR</b>	<b>EquNA</b>	<b>EquAsP</b>	<b>EquEM</b>	<b>RateEUR</b>	<b>RateUSD</b>	<b>RateJPY</b>	<b>CredSta</b>	<b>CredSwa</b>
=> NWE volatility: optimal BW	33	33	33	33	33	33	33	33	33	33
=> innovation fitting:										
m-	46.2509	29.8744	44.5616	12.0588	88.9649	21.3967	26.0127	15.3459	5.5264	24.4995
c-	9.3506	7.4452	9.3763	4.8388	14.7641	6.0432	7.2913	4.6700	2.6991	6.6741
m+	n.a.	n.a.	38.6491	n.a.	11.4168	7.1200	9.9004	97.3716	3.2560	11.9297
c+	n.a.	n.a.	8.1619	n.a.	3.6530	3.3999	3.8580	15.7242	1.8623	4.5008
<b>One-sided setup</b>										
=> NWE volatility: optimal BW	47	47	47	47	47	47	47	47	47	47
=> innovation fitting:										
m-	13.7634	10.6318	16.0473	8.0682	10.5873	185.1455	18.0548	33.4894	5.7823	19.9826
c-	5.2129	4.4950	5.6785	3.8099	4.5792	18.5787	5.7340	7.6884	2.8790	6.1044
m+	n.a.	n.a.	30.7702	214.6806	119.6865	7.4508	12.6285	6.3764	3.7422	15.8454
c+	n.a.	n.a.	7.2493	19.6216	14.2799	3.5334	4.6717	3.1765	2.1143	5.3567
<b>Two-sided setup</b>	<b>CredAAA</b>	<b>CredAA</b>	<b>CredA</b>	<b>CredBBB</b>	<b>CredEM</b>	<b>CredHY</b>	<b>CurrEUR</b>	<b>CurrGBP</b>	<b>CurrCHF</b>	<b>CurrSEK</b>
=> NWE volatility: optimal BW	33	33	33	33	33	33		33	33	33
=> innovation fitting:										
m-	19.2648	17.8981	7.0803	4.9753	3.8879	6.2181		22.7389	77.9770	n.a.
c-	5.8810	5.6569	3.2452	2.6175	1.9129	2.6415		7.1687	11.9471	n.a.
m+	6.3485	6.0744	14.0085	4.3502	7.4171	7.2583		n.a.	9.7411	19.0329
c+	3.0205	2.9544	4.9699	2.2894	3.6985	3.7305		n.a.	4.0773	5.4981
<b>One-sided setup</b>										
=> NWE volatility: optimal BW	47	47	47	47	47	47		47	47	47
=> innovation fitting:										
m-	13.4074	17.7453	8.4912	5.6594	6.0792	10.1990		12.5178	76.6404	37.7481
c-	5.0211	5.7718	3.7544	2.7274	2.8457	3.9446		4.8799	11.6961	8.6673
m+	5.6672	5.4843	7.2931	4.9523	5.0484	6.0966		n.a.	7.5823	n.a.
c+	2.8178	2.7785	3.4011	2.7361	2.7651	3.1689		n.a.	3.6632	n.a.
<b>Two-sided setup</b>	<b>CurrDKK</b>	<b>CurrNOK</b>	<b>CurrUSD</b>	<b>CurrCAD</b>	<b>CurrJPY</b>	<b>CurrAUD</b>	<b>CurrNZD</b>	<b>CurrSGD</b>	<b>CurrHKD</b>	<b>CurrEM</b>
=> NWE volatility: optimal BW	33	33	33	33	33	33	33	33	33	33
=> innovation fitting:										
m-	5.6675	9.6028	18.5928	12.3762	27.5958	9.8600	9.4686	n.a.	n.a.	n.a.
c-	2.7300	4.3296	6.3114	5.0882	6.9808	4.6341	4.3857	n.a.	n.a.	n.a.
m+	4.1212	15.0128	34.5563	132.7842	10.3020	8.6226	8.2711	21.8849	n.a.	n.a.
c+	2.0895	4.5555	7.4494	14.5583	4.2345	3.1670	3.1810	5.9271	n.a.	n.a.
<b>One-sided setup</b>										
=> NWE volatility: optimal BW	47	47	47	47	47	47	47	47	47	47
=> innovation fitting:										
m-	4.6265	7.3635	17.1833	10.8337	47.6895	6.0316	7.6501	45.7300	n.a.	88.9839
c-	2.2949	3.5614	5.7187	4.4140	9.2340	3.1715	3.6597	9.4788	n.a.	13.3509
m+	4.4370	35.7130	140.6146	n.a.	10.2925	17.1714	13.6044	58.3708	809.1065	n.a.
c+	2.2817	7.9004	16.0605	n.a.	4.3242	5.1772	4.6098	10.4077	39.0338	n.a.

Figure 15: Parameters for the non-stationary modeling (2) of daily benchmark returns from 1999 to 2008, based on wholistic smoothing of the covariance matrix and with reference to the two-sided or the one-sided implementation. Gray highlighted elements designate manually adjusted bandwidths after CV. Cells with entry 'n.a.' identify series where the Pearson VII innovation fitting failed.

In addition, the interrelationship between nonparametric volatility estimation and the Pearson VII fitting of innovations is exemplified with the return series *EquNA*, *RateUSD*, *CredSta* and *CurrUSD* in figures 16

till 19.<sup>26</sup> This makes round our representative extract of the full non-stationary setup (2). The *EquNA* sample reflects the equity market crises in the period 1999 to 2008. Significant peaks and an increased volatility are observed in the early new millennium at the 'burst of the I.T. bubble' and the economic crisis until 2003. After even-tempered years of equity recoveries, volatility increased with the beginning Subprime-crisis in 2007 and jumped in September 2008 (collapse of US investment banks) dramatically from the long-term average of 18% to temporarily more than 75% p.a. in terms of nonparametric estimates. The NWEs immediately react on sequences of extreme log-returns and detect phases of high and low volatility, while the empirical standard deviation (258-day) takes long until extreme changes get an impact. The innovation distribution is heavy-tailed, where the asymmetric Pearson VII density fits the residual histogram well.

Regarding the diff-returns and the nonparametric volatility of *RateUSD* we exhibit four phases. Lower dynamics are assessed within the first two years and the intermediate years 2005 - 2006. Shocks in interest rates accompany the economic crisis from 2001 to 2003, where 5-year US swaps declined from more than 7% to a low of 2.5% p.a.; parallel the annualized volatility moved from a level of 60bp to about 180bp. With the Subprime-/ financial crisis the volatility increased again sharply from 55bp (minimum in May 2007) to 248bp at its maximum in October 2008. Again the Pearson VII fit of innovations works (visually) excellent. The spread dynamic *CredSta* develops similarly the last two years, where the swap spread decreases from -25bp to -102bp (on September 18, 2008, right after Lehman Brothers' insolvency) due to investors flight into the safe haven of treasuries. Consequently, NWEs react with an annualized volatility of more than 108bp, whereas the average volatility was only 14bp from 2003 to 2006. The innovation density is very heavy-tailed (e.g.  $f_{3,26,1.86,5.53,2.70}^{VII}$  for the two-sided approach) as a closer inspection of the histogram and its approximation in figure 18 reveals. The exchange rate EUR to USD varies most time in a range from 7% to 13% annualized volatility. Exceptions are increased dynamics in the years 2000 to 2002, a low volatility of about 5% in summer 2007, and an extreme jump to more than 20% p.a. at the end of 2008, where the USD was revalued about 28% against the EUR within two months. Regarding the whole period 1999 to 2008 the innovations are skewed to the left and the Pearson VII fit enables this asymmetry.

Concluded from the preceding four figures, from the time-dependent correlation estimation of this examples (figure 12) and the results of the complete model approximation regarding the 30 benchmark indices (figure 15), the wholistic non-stationary approach turns out to be flexible and convenient in modeling the multivariate return dynamics.

## 4.2 Forecasting experiment

Finally, we aim at predicting (interrelated) future returns of financial instruments or entire portfolios with the non-stationary regression model. To assess the forecasting quality, we divide the daily return series into two subsamples: the 'in-sample part' from 1999 to 2006 (2061 return observations) is used to calibrate the multivariate return model (2) in terms of the one-sided setup;<sup>27</sup> the 'out-of-sample part' from 2007 to 2008 (sample size 515) applies the model to specify future return distributions and to test them against realized

<sup>26</sup>Volatility series start at February 18, 1999 to reduce boundary errors (date point 33 according to  $\bar{h}^{opt}$ ).

<sup>27</sup>The model is calibrated once in-sample and the obtained parameters (bandwidth for MNWE, Pearson VII parametrization or quantiles) are fixed in the out-of-sample analysis. Alternatively one could recalibrate the model every day or with fixed period.



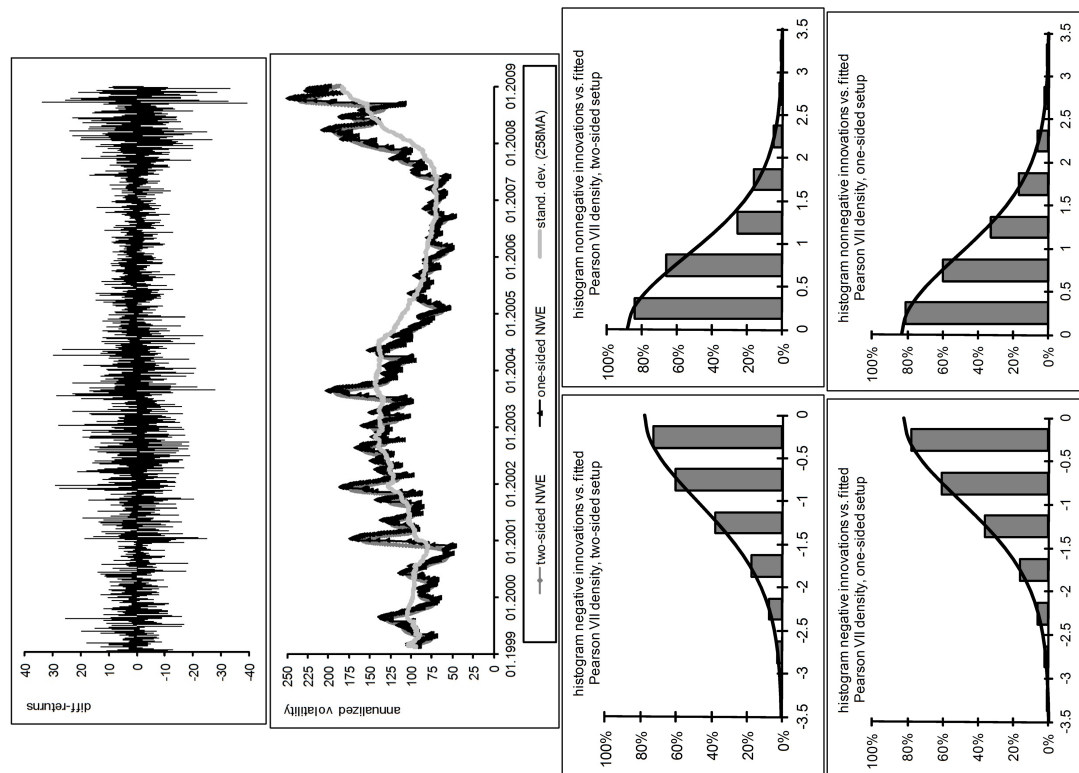


Figure 16: Top: Daily log-returns of *EquiNA* from 1999 to 2008. Middle: Nonparametric volatility estimates  $\hat{\sigma}_t(t)$  compared to standard deviation. Bottom: Left and right Pearson VII fit  $f_{m,c}^{VII(1)}$  of innovations for the two- and one-sided approach. For used parametrization see figure 15.

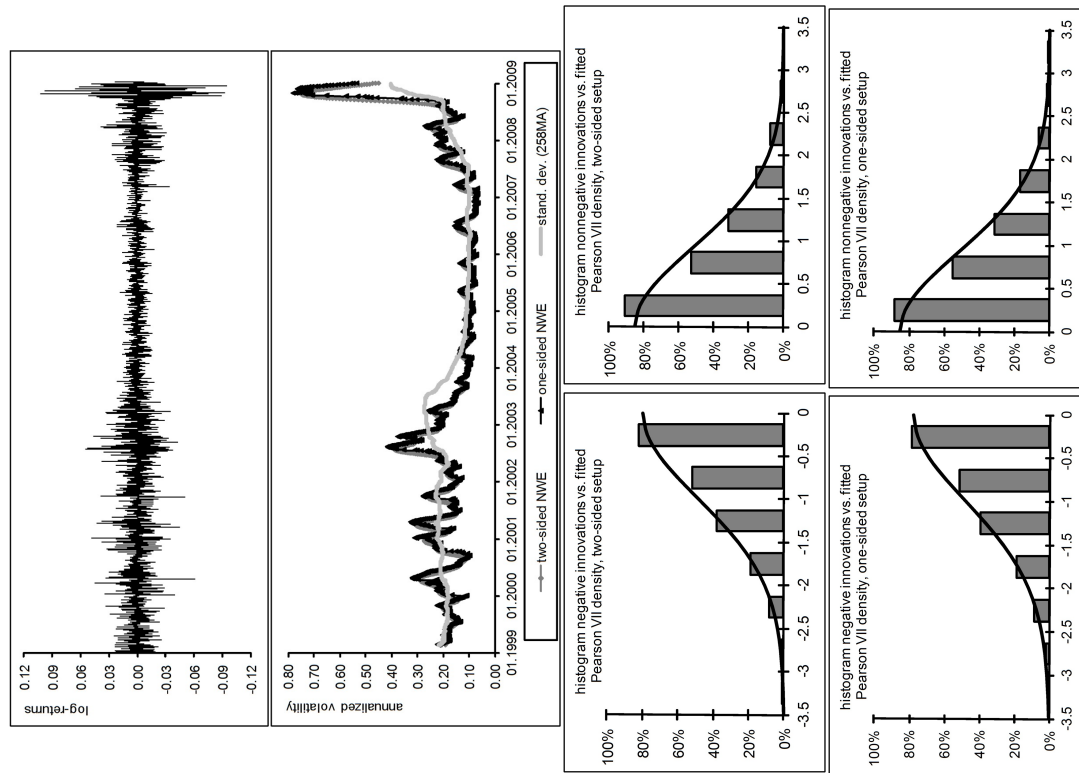


Figure 17: Top: Daily diff-returns of *RateUSD* 1999 to 2008. Middle: Nonparametric volatility estimates  $\hat{\sigma}_t(t)$  compared to standard deviation. Bottom: Left and right Pearson VII fit  $f_{m,c}^{VII(1)}$  of innovations for the two- and one-sided approach. For used parametrization see figure 15.

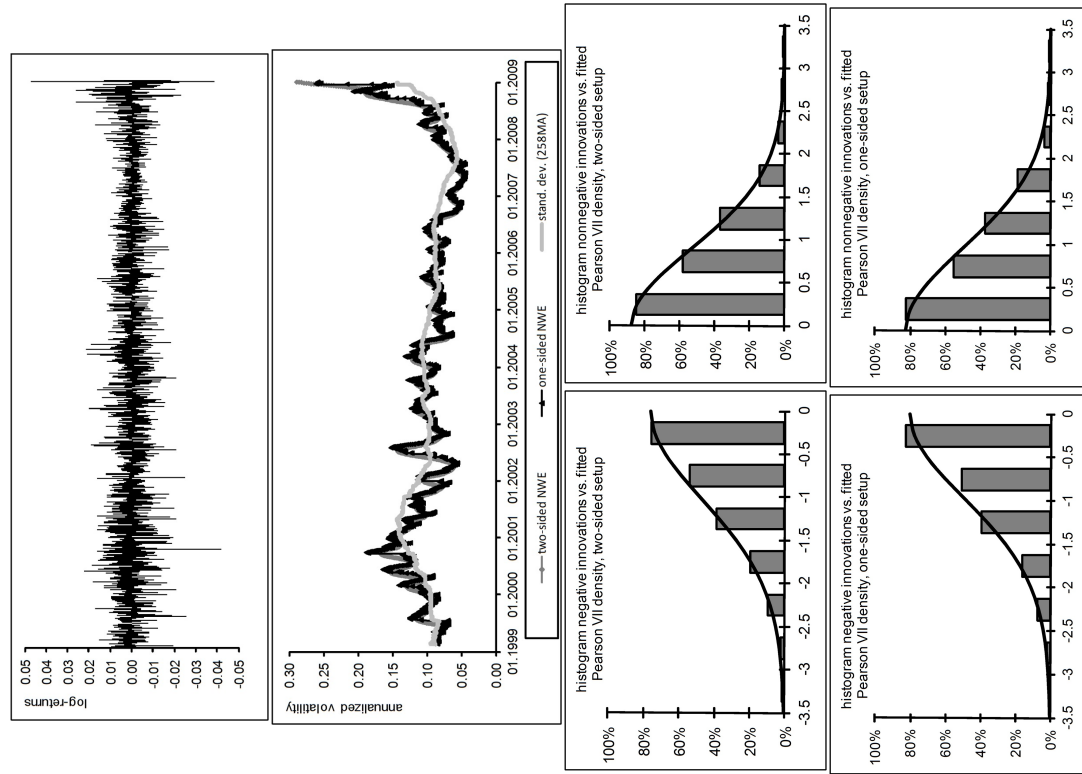


Figure 19: Top: Daily log-returns of *CurrUSD* 1999 to 2008. Middle: Nonparametric volatility estimates  $\hat{\sigma}_t(t)$  compared to standard deviation. Bottom: Left and right Pearson VII fit  $f_{m, \cdot}^{VII(1)}$  of innovations for the two- and one-sided approach. For used parametrization see figure 15.

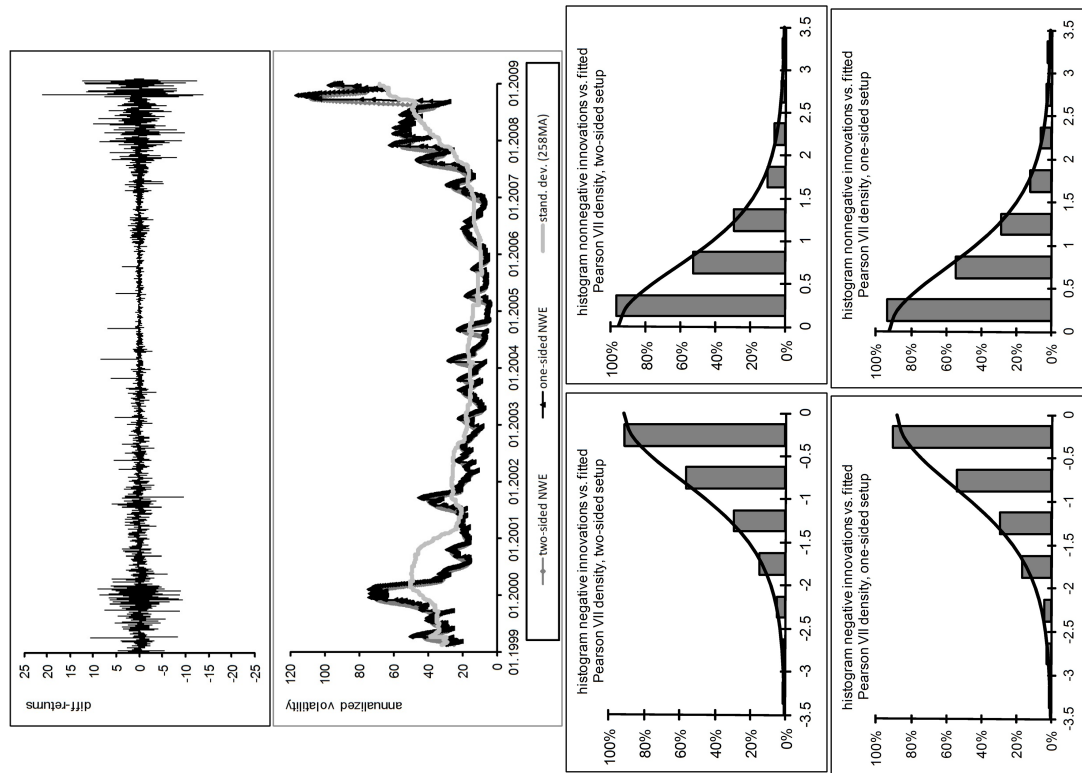


Figure 18: Top: Daily diff-returns of *CredSta* 1999 to 2008. Middle: Nonparametric volatility estimates  $\hat{\sigma}_t(t)$  compared to standard deviation. Bottom: Left and right Pearson VII fit  $f_{m, \cdot}^{VII(1)}$  of innovations for the two- and one-sided approach. For used parametrization see figure 15.

(portfolio) returns. Beginning with  $t_0 = 2061$  and the information available in  $t \geq t_0$ , we forecast the distribution of the 1-day ahead return vector  $\mathbf{X}_{t+1}$  and its portfolio return  $W_{t+1}$ , successively.

Our exemplary asset portfolio is structured as a 'classical risky' but diversified mix consisting of 30% equity investments and 70% debt securities. The equity allocation is oriented on a regional structured MSCI World index (Developed Markets) added by MSCI Emerging Markets. We predefine an arbitrary constant regional weighing and deduce equity exposures and currency exposures from the MSCI index compositions on December 29, 2006. The debt allocation is concentrated on investment grade (bullet) bonds of the Eurozone, USA and Japan, benchmarked by Merrill Lynch Broad Market indices, with an admixture of High Yield bonds (ML US HY) and Emerging Market bonds (JP Morgan EMBI+). Again we predefine weights of economic areas and deduce interest rate- and credit spread exposures from real index compositions at the same record date, but we adjust the modified duration class-wise to 5%.<sup>28</sup> Our portfolio structure and the resulting exposure vector  $(\omega_1, \dots, \omega_{30})$  for an investment volume of 1 billion EUR (from the perspective of an EUR-investor) are presented in figure 20.

<b>Equity</b>	<b>30%</b>	<b>EquEUR</b>	59,100,000
Europe	40%	<b>EquexEUR</b>	60,900,000
North America	40%	<b>EquNA</b>	120,000,000
Asia/ Pazific	15%	<b>EquAsP</b>	45,000,000
Emerging Markets	5%	<b>EquEM</b>	15,000,000
<b>Debt</b>	<b>70%</b>	<b>RateEUR</b>	-140,000
EUR	40%	<b>RateUSD</b>	-175,000
USA*	50%	<b>RateJPY</b>	-35,000
JPY	10%	<b>CredSta</b>	-83,650
		<b>CredSwa</b>	-54,950
		<b>CredAAA</b>	-10,150
		<b>CredAA</b>	-38,500
		<b>CredA</b>	-71,400
		<b>CredBBB</b>	-56,350
		<b>CredEM</b>	-17,500
		<b>CredHY</b>	-17,500
		<b>CurrEUR</b>	759,100,000
		<b>CurrGBP</b>	41,100,000
		<b>CurrCHF</b>	12,000,000
		<b>CurrSEK</b>	4,500,000
		<b>CurrDKK</b>	1,500,000
		<b>CurrNOK</b>	1,500,000
		<b>CurrUSD</b>	111,900,000
		<b>CurrCAD</b>	8,100,000
		<b>CurrJPY</b>	32,700,000
		<b>CurrAUD</b>	8,100,000
		<b>CurrNZD</b>	300,000
		<b>CurrSGD</b>	1,500,000
		<b>CurrHKD</b>	2,700,000
		<b>CurrEM</b>	15,000,000

\* incl. HY 5% and EM 5%

Figure 20: Exemplary portfolio structure regarding the 'classical risk portfolio' and the aggregated exposure vector for an investment of 1 billion EUR. The description of exposure classes is provided in table 1.

We assume a compensation of all gains and losses at the end of each trading day (without transaction costs), i.e. the portfolio is rebalanced and the exposures  $\omega$  are constant over the course of time. Hence, the portfolio return  $W_{t+1}$  on the horizon  $(t, t + 1]$  is to be calculated as:

$$W_{t+1} = \sum_{j=1}^{30} \omega_j X_{j,t+1} = \omega' \mathbf{X}_{t+1}. \quad (25)$$

<sup>28</sup>Moreover, we presume a perfect currency hedge on the bond side. Equity investments are modeled currency unhedged.

Coming back to the in-sample calibration, that is independently from exposure weights, we first assess in terms of PNWEs  $\hat{\sigma}_{(1)k,l}^2(t)$  ( $k, l = 1, \dots, d$ ) the matrix  $\mathbf{h}_{(1)}^{\text{opt}}$  of optimal bandwidths (with  $h_{(1)k,l}^{\text{opt}}$  occasionally adjusted with respect to corresponding volatility estimates). By reason of the need for a wholistic bandwidth, the average of individual bandwidths over the matrix diagonal is built (compare section 3), leading to a global optimum  $\bar{h}_{(1)}^{\text{opt}} = 76$  days.<sup>29</sup> The one-sided nonparametric estimation  $\hat{\Sigma}_{(1)}^2(t)$  for the covariance matrix is executed, its positive semidefiniteness is tested successfully outside the boundary regions. The in-sample series of estimated innovations  $\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_{t_0}$  is fitted component-wise by the asymmetric Pearson VII density  $f_{\varepsilon_k; m_+, c_+, m_-, c_-}^{\text{VII}}$ , whose parametrization is listed in figure 21. The Pearson VII fit fails only five times and each on the right tail. Compared to the full sample approximation the failure rate is reduced marginally (with a new fault at *CurrUSD*), but the tail index point estimate is reduced in all samples due to the smoother covariance function. The majority of series turn out to be quite heavy tailed. Using the normal density as a substitute for failed Pearson VII tails, the distribution of returns  $\mathbf{X}_t$  is specified completely.

Parameters for complete implementation of the non-stationary model										
One-sided setup	EquEUR	EquexEUR	EquNA	EquAsP	EquEM	RateEUR	RateUSD	RateJPY	CredSta	CredSwa
=> NWE volatility:										
CV-optimal BW	76	76	76	76	76	76	76	76	76	76
=> innovation fitting:										
m-	7.9085	5.7935	13.7365	4.9903	6.0140	19.8037	9.9209	9.7034	4.6532	10.0848
c-	3.7423	3.0567	5.0516	2.7358	3.2300	5.7820	4.0549	3.8635	2.3548	4.0928
m+	n.a.	n.a.	10.7769	12.1557	38.9318	5.6374	7.2073	4.3640	3.0543	7.1788
c+	n.a.	n.a.	4.1317	4.4472	7.9601	2.9843	3.3615	2.4791	1.7649	3.3906
One-sided setup	CredAAA	CredAA	CredA	CredBBB	CredEM	CredHY	CurrEUR	CurrGBP	CurrCHF	CurrSEK
=> NWE volatility:										
CV-optimal BW	76	76	76	76	76	76	76	76	76	76
=> innovation fitting:										
m-	8.0532	9.0355	5.9813	4.2747	4.0071	8.2513		6.3705	12.3409	10.5037
c-	3.7268	3.9855	3.0338	2.2304	2.0877	3.4821		3.2238	4.4294	4.2925
m+	4.4105	4.5755	6.2168	4.0143	4.1816	4.4683		n.a.	4.5282	23.7482
c+	2.2922	2.3573	2.9503	2.2816	2.3595	2.5089		n.a.	2.5601	6.5016
One-sided setup	CurrDKK	CurrNOK	CurrUSD	CurrCAD	CurrJPY	CurrAUD	CurrNZD	CurrSGD	CurrHKD	CurrEM
=> NWE volatility:										
CV-optimal BW	76	76	76	76	76	76	76	76	76	76
=> innovation fitting:										
m-	4.2683	4.9823	7.2926	6.7882	7.8773	6.2219	6.4975	26.8540	35.3213	20.1484
c-	2.3279	2.7435	3.5171	3.3551	3.4597	3.2002	3.2978	7.2020	8.3856	6.1695
m+	4.2663	29.9596	n.a.	n.a.	6.8593	11.9756	10.9255	23.7653	20.8520	27.9213
c+	2.4410	7.0998	n.a.	n.a.	3.2487	4.1852	4.0272	6.4401	5.9625	6.9801

Figure 21: In-sample calibration of the non-stationary modeling (2) regarding daily benchmark returns from 1999 to 2006 (one-sided setup, wholistic smoothing of covariance matrix). Cells with entry 'n.a.' identify series where the Pearson VII innovation fitting failed.

To assess the modeling performance, we apply the Kupiec test to shortfall rates of the out-of-sample part. The two-sided hypothesis test is an extension of a binomial test for the likelihood of  $N$  shortfalls in a sample of size  $n$ , where the true shortfall probability is hypothetical  $H_0 : p = \alpha$  for a  $(1 - \alpha)$  VaR-level. Based on a normal approximation, Kupiec (1995) developed approximate 95% confidence regions of failure rates. The log-likelihood-ratio

$$\text{LR}_{n,p}(N) = -2 \ln [(1-p)^{n-N} p^N] + 2 \ln \left[ \left(1 - \frac{N}{n}\right)^{n-N} \left(\frac{N}{n}\right)^N \right] \quad (26)$$

is  $\chi_1^2$ -distributed under  $H_0$ . Thus, the risk measure and  $H_0$  are rejected on a 5% level of significance if  $\text{LR}_{n,p}(N) > 3.84$ . The Kupiec backtesting is widely used to evaluate risk models theoretically, amongst

<sup>29</sup>This differs somewhat from the full sample optimal bandwidth with a tendency to oversmoothing, due to missing extreme realizations within the last two years.

others in Choi and Nam (2008), or practically, as the penalty zones of the Basel II committee are based on this methodology (see e.g. Jorion (2006)).

The regression model implementation of the 30-dimensional return process is backtested via quantile forecasts  $\text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}(t)$  regarding the next day's portfolio return against realized portfolio returns  $W_{t+1}$ . The estimate for a maximum loss, that is not exceeded with probability  $1 - \alpha$  (1-day Value at Risk), can be derived analytically in terms of the non-stationary model:<sup>30</sup>

$$\text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}(t) = \boldsymbol{\omega}' \bar{\mathbf{X}}_{t-1} - \sqrt{\left( \boldsymbol{\omega} \odot \mathbf{u}_{m-, c-, \alpha^{(1)}} \right)' \hat{\boldsymbol{\sigma}}_{(1)}(t) \hat{\boldsymbol{\rho}}_{(1)}(t) \hat{\boldsymbol{\sigma}}_{(1)}(t) \left( \boldsymbol{\omega} \odot \mathbf{u}_{m-, c-, \alpha^{(1)}} \right)} \quad (27)$$

where  $\boldsymbol{\rho}(t) := \boldsymbol{\sigma}^{-1}(t) \boldsymbol{\Sigma}^2(t) \boldsymbol{\sigma}^{-1}(t)$  and  $\boldsymbol{\sigma}(t) := \sqrt{\text{Diag}(\boldsymbol{\Sigma}^2(t))}$ ,

and  $u_{m-, c-, \alpha^{(1)}}^{\varepsilon k}$  is the  $\alpha^{(1)} = 2\alpha$  quantile of the left-sided Pearson VII fit (of the  $k$ -th component).<sup>31</sup> A shortfall is observed if  $W_{t+1} \leq \text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}(t)$ .

The out-of-sample execution of the forecasting experiment is sobering at first glance: Concerning the 99% confidence level the portfolio returns entail 14 exceedances over the VaR threshold, which deviates significantly from the expectation 5.2. The upper limit of permitted shortfalls was 10 following a 5% significance level of the Kupiec test, and the non-stationary hypothesis is denied; figure 22 summarizes. Regarding other levels of (analytically derived) maximum portfolio losses the regression approach is accepted by backtesting the 80% and the 99.9% VaR. It is rejected for confidence levels from 90% to 99.5% and at the extreme tail of 99.95%. As exhibited in the upper tableau of figure 23, the number of exceedances against model forecasts is above the expected quantity in each case, but for usual confidence levels from 95% to 99.5% the counted shortfalls are only marginally greater than the boundaries of the acceptance range.

LR test (Kupiec-test), 5% level	
VaR confidence level (as $1-\alpha$ )	99.00%
observations (sample size)	515
number of VaR exceedances	14
shortfall probability p	1.0%
expected exceedances of VaR	5.2
significance level of LR-Test (as $1-\beta$ )	95%
c-value $\chi^2$ -distribution (one-sided)	3.84
test statistic (LR)	10.46
	10.46 > 3.84
Non-stationary model acceptance	no

Figure 22: Kupiec backtesting of forecasted  $\text{VaR}_{W, \text{RM}}^{99\%, 1d}(t)$  against realized portfolio returns  $W_{t+1}$  (out-of-sample) in the non-stationary modeling (2) of the multivariate return distribution of the 'classical risk portfolio' (equity : debt = 30 : 70)

We compare the goodness of fit of the non-stationary model with a delta-normal model, working with Gaussian quantiles and empirical estimates. The corresponding parametric VaR has the following form:

<sup>30</sup>We proved that the left tail is the (maximum) loss tail for the overall portfolio and its risk interdependencies (despite losses for interest rate- and credit spread exposures are on the right tail), since the downside potential of equities is predominant.

<sup>31</sup>The connection to the scaled Student- $t$  quantile is utilized:  $u_{m, c; \alpha} = \frac{c}{\sqrt{2m-1}} t_{2m-1; \alpha}$ .

VaR confidence level (as $1-\alpha$ )	number of exceedances, model rejection			
	expectation	upper limit	non-stationary model	normal model
80.00%	102.6	121	120	137
90.00%	51.3	65	77	96
95.00%	25.7	35	40	61
98.00%	10.3	17	18	38
98.50%	7.7	13	15	33
99.00%	5.1	10	14	29
99.50%	2.6	6	7	23
99.90%	0.5	2	2	14
99.95%	0.3	1	2	9

VaR confidence level (as $1-\alpha$ )	number of exceedances, model rejection			
	expectation	upper limit	non-stationary model	normal model
80.00%	102.6	121	116	138
90.00%	51.3	65	73	95
95.00%	25.7	35	36	62
98.00%	10.3	17	17	38
98.50%	7.7	13	15	37
99.00%	5.1	10	14	29
99.50%	2.6	6	8	22
99.90%	0.5	2	2	11
99.95%	0.3	1	2	8

Figure 23: Kupiec test of the non-stationary model and the delta-normal model on several levels of confidence. Rejections on a 5% level of significance are gray highlighted. Top: evaluation of portfolio forecasts from the analytical implementation; bottom: forecast evaluation from the simulation approach.

$$\text{VaR}_{W,PM}^{1-\alpha,1d}(t) = \omega' \bar{X}_{t-1} + z_\alpha \sqrt{(\omega \odot \bar{\sigma}_t)' \bar{\rho}_t (\omega \odot \bar{\sigma}_t)}, \quad (28)$$

where the standard normal  $\alpha$ -quantile  $z_\alpha$  is a scalar and  $\bar{\sigma}_t$ ,  $\bar{\rho}_t$  are the empirical standard deviation vector and correlation matrix, based on the last 258 returns. We backtest its return forecasts the same way and compare the model performances in figure 23 (upper tableau concerning portfolio losses exceeding the analytical VaR bound). This basic risk model causes substantially more shortfalls and is rejected on the 5% level of significance for all confidence levels  $1 - \alpha \geq 80\%$ . Moreover, for risk levels greater than 95% the observed exceedances of parametric VaR are more than double the upper limit of acceptance and also more than double the shortfall number experienced to the non-stationary forecasts. The latter regression approach dominates (weakly) the delta-normal model.

The analytical VaR estimates and the return distributions of both models are validated by a simulation approach, additionally. Based on the (one-sided) mean return, the nonparametric covariance matrix  $\hat{S}_t \hat{S}_t' = \hat{\Sigma}_{(1)}(t) = \hat{\sigma}_{(1)}(t) \hat{\rho}_{(1)}(t) \hat{\sigma}_{(1)}(t) = \hat{\sigma}_{(1)}(t) \hat{L}_{(1)}(t) \hat{L}_{(1)}'(t) \hat{\sigma}_{(1)}(t)$  (using Cholesky decomposition) and the specified innovation distribution, a multivariate return vector  $\tilde{X}_t$  is simulated via random innovations  $\tilde{\varepsilon}_t$ :

$$\tilde{X}_t = \bar{X}_{t-1} + \hat{S}_t \tilde{\varepsilon}_t = \bar{X}_{t-1} + \hat{\sigma}_{(1)}(t) \hat{L}_{(1)}(t) \tilde{\varepsilon}_t. \quad (29)$$

We implement a Monte-Carlo simulation of portfolio returns  $\tilde{W}_t = \omega' \tilde{X}_t$  to assess the  $\text{VaR}_{W,RM}^{1-\alpha,1d}(t)$  regarding the next day's return  $W_{t+1}$  in each point  $t \geq t_0$  out-of-sample. By dint of 10,000 simulation runs the VaR is read off the order statistic  $\tilde{W}_t^{(1)}, \dots, \tilde{W}_t^{(10,000)}$  of simulated returns (e.g.  $\tilde{W}_t^{(100)}$  corresponds

to the 99% VaR projected from  $t$ ).<sup>32</sup> Again we contrast VaR forecasts of the delta-normal model with an analogous simulation.<sup>33</sup> The results are displayed at the bottom tableau of figure 23.

Regarding the non-stationary model, up to 4 shortfalls less are observed for confidence levels lower 98% (but one more exceeding at the 99.5% level). As a result our approach is accepted also for the 98% VaR-level, where the decrement to absolutely 17 shortfalls goes along with the upper limit of the Kupiec acceptance band. There is no systematic improvement of the simulation against the analytical approach, although the full return distribution (not only the left tail of portfolio returns) is involved. Deviations might be explained by the limited number of simulation runs.<sup>34</sup> Accordingly, for 3 of 9 tested confidence levels the non-stationary simulation implementation is successfully tested. At the delta-normal simulation the shortfall differences to the analytical results vary in a range from  $-3$  to  $+4$ , but they have no influence on the Kupiec decision. The observed exceedances deviate still significantly from the expectation and parametric VaRs are rejected on all levels. Our regression model clearly outperforms the delta-normal model. Overall the non-stationary modeling regarding financial returns from 2007 to 2008 (calibrated on 8-year daily returns before) and the exemplary portfolio is to be graded as (still) sufficient. Its analytical and simulative VaR forecasts cause shortfall numbers close to the boundaries of the Kupiec test, even though a historical (new) market phase accompanied by dramatic shocks in equity markets and interest rates is run through and the in-sample frame suggested a comparatively smooth calibration.

In addition we check the model approximation and the forecasting abilities for three variations of the asset allocation: 1. a 'conservative risk portfolio' with an equity ratio of 25%, 2. an 'aggressive risk portfolio' with 50% equities, 3. an 'overall portfolio' being representative for life insurance with a major weight on fixed income (10% equity). The forecasted return distributions from the non-stationary implementation (vs. the delta-normal model) are tested again with Kupiec for several confidence levels. The results following analytical VaR estimates are displayed in figure 24. Concerning the first variation, the number of shortfalls  $W_{t+1} \leq \text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}(t)$  increases by the majority, even the 80% VaR is now rejected and the non-stationary model is accepted only for the 99.9% confidence level. Otherwise the delta-normal approximation of return dynamics is rejected again for all levels with significantly too much outliers. The second 'aggressive' portfolio improves the outcomes of the non-stationary modeling. Although slightly more shortfalls are observed within 95% to 98.5% levels, its number is reduced for the other return quantiles. Especially the extreme tail is approximated well, resulting in acceptance for VaR-levels from 99.5%. Again the normal model fails significantly for all confidence levels. In modeling dynamics of the 'overall portfolio' the non-stationary success is heterogenous. A better approximation towards the distribution's middle is observed, where exceedances are close enough to the expectation for quantiles  $\alpha > 5\%$ . However, the tail approximation worsens with significantly too large shortfall numbers (except the extreme tail 99.95%). Repeatedly the parametric VaR is outperformed clearly, although its test results are still best for the last portfolio.

<sup>32</sup>In detail and due to the sake of simplicity we work first with 1,000 runs (except for the 99.95% confidence level that requires at least 2,000), but reproduce with 10,000 runs at each 'bottleneck' point  $t$  where  $\left| \frac{\text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}(t) - W_{t+1}}{\text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}} \right| < \delta_{1-\alpha}$ . We derived  $\delta_{1-\alpha} \equiv \delta := 12\%$  as an adequate constant from the standard deviation of VaR-forecasts on the 1,000s simulations.

<sup>33</sup>Here simulated returns follow as  $\tilde{\mathbf{X}}_t = \tilde{\mathbf{X}}_{t-1} + \tilde{\sigma}_t \odot \tilde{\mathbf{L}}_t \tilde{\epsilon}_t$ , where  $\tilde{\mathbf{q}}_t := \tilde{\mathbf{L}}_t \tilde{\mathbf{L}}_t'$  and  $\epsilon_{k,t} \sim \mathcal{N}(0, 1)$  iid in time and coordinates.

<sup>34</sup>Furthermore the exertion of Cholesky decomposition of correlation matrices causes smaller errors, since it is exact only for multivariate normal returns.

Portfolio weighting equity : debt = 25 : 75

VaR confidence level (as $1-\alpha$ )	number of exceedances, model rejection			
	expectation	upper limit	non-stationary model	normal model
80.00%	102.6	121	122	134
90.00%	51.3	65	69	88
95.00%	25.7	35	44	62
98.00%	10.3	17	18	38
98.50%	7.7	13	17	30
99.00%	5.1	10	13	26
99.50%	2.6	6	10	21
99.90%	0.5	2	2	11
99.95%	0.3	1	2	10

Portfolio weighting equity : debt = 50 : 50

VaR confidence level (as $1-\alpha$ )	number of exceedances, model rejection			
	expectation	upper limit	non-stationary model	normal model
80.00%	102.6	121	113	131
90.00%	51.3	65	74	94
95.00%	25.7	35	41	65
98.00%	10.3	17	21	40
98.50%	7.7	13	18	38
99.00%	5.1	10	11	33
99.50%	2.6	6	4	24
99.90%	0.5	2	1	10
99.95%	0.3	1	1	9

Portfolio weighting equity : debt = 10 : 90

VaR confidence level (as $1-\alpha$ )	number of exceedances, model rejection			
	expectation	upper limit	non-stationary model	normal model
80.00%	102.6	121	110	122
90.00%	51.3	65	62	72
95.00%	25.7	35	36	45
98.00%	10.3	17	23	30
98.50%	7.7	13	18	25
99.00%	5.1	10	15	23
99.50%	2.6	6	10	19
99.90%	0.5	2	5	12
99.95%	0.3	1	1	11

Figure 24: Kupiec tests of return forecasts from the non-stationary model and the normal risk model on several levels of confidence. Rejections on a 5% level of significance are gray highlighted. The approximation quality is tested for three more portfolios; top: 'conservative risk portfolio' (equity ratio 25%); middle: 'aggressive risk portfolio' (equity ratio 50%); bottom: 'overall portfolio repr. insurance' (equity ratio 10%).

Concluding, the non-stationary model is able to replicate multivariate return dynamics and to provide satisfactory forecasts. The combination of a wholistic smoothing conception for nonparametric covariances and asymmetric, heavy-tailed innovations works well and consistently. The regression approach outperforms a basic Gaussian risk model, although the chosen time frame of real financial markets and the risky portfolio structure were challenging. Following the broad empirical study, our model implementation turned out to be a flexible, manageable and intuitive approach for modeling non-stationary financial returns.



## 5 Summary and conclusion

In this article we motivate and analyze a multivariate non-stationary model for the dynamics of financial returns. Based on the idea that recent past and future returns depend on the same unknown economic factors, evolving smoothly through time and manifested in pricing, the current level of (co-)variance is dominated exogenously. In a multiplicative approach following Herzel et al. (2005), the return vectors  $\mathbf{X}_t \in \mathbb{R}^d$  are assumed to have a time-varying unconditional covariance matrix  $\Sigma^2(t)$  that is modeled via classical nonparametric regression (Nadaraya-Watson estimates, MNWE). The standardized residuals are modeled parametrically with the asymmetric Pearson type VII distribution. With reference to Gürtler et al. (2009), we outline statistical results as consistency of a two-sided MNWE and of a one-sided MNWE (based only on past data) for the covariance matrix. The further distinction between wholistic smoothing, working with a global optimal bandwidth, and individual smoothing, working with individual bandwidths for covariance pairs, is fundamental. Although the pairwise nonparametric estimates are also consistent, the composed matrix is not, since its positive (semi)definiteness is endangered due to the different degrees of smoothing.<sup>35</sup> Hence, wholistic smoothing is the final practical solution and for forecasting purposes the one-sided (historical) implementation is applied. Rules for selecting optimal bandwidths, that are originally based on cross-validation of pairwise estimates, are provided. The distributional fit of (estimated) innovations works component-wise, the task is simplified by providing a method of moments for Pearson VII parameter estimation and a connection to the Student- $t$  distribution. A factor-based (analytical) VaR calculation in terms of the non-stationary regression model is enabled.

The main focus of our paper pertains multivariate empirical return dynamics, exemplified by a 30-dimensional setup of equity indices, interest rates, credit spreads and exchange rates, that are recorded daily from January 1999 to December 2008. Univariate statistical tests on normality, serial identity and independence of those time series are already executed in Gürtler and Rauh (2009), leading to clear rejections and conclusions of (unconditional) heteroscedasticity and non-stationarity. In the present paper we analyze initially the behavior of linear dependence of the financial series, i.e. the development of covariance and correlation estimates over time. By dint of empirical estimators (and the knowledge of time-varying volatilities) not only the covariance- but also the correlation structures change significantly between market phases. With Fisher's  $z$ -test we reject time-invariant correlations especially for daily returns. The window length, periodicity and weighting of included data play an important role for estimation. The motivation of a non-stationary (nonparametric) model for risk interdependence and multivariate financial returns is established.

In terms of our benchmark universe we survey the characteristics of nonparametric covariance estimates, starting with the individual smoothing conception. We notice that bandwidth selection can not be automatized at all, but we provide other quantitative criteria. Moreover, we observe a trade-off between volatility estimation and innovation fitting (the bigger the bandwidth of volatility estimates the more heavy tailed are the innovations), that might cause failing the Pearson VII approximation if a kurtosis lower 3 is realized; then the Gaussian distribution is a conservative alternative. Graphs of covariances and their standardization

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<sup>35</sup>The problem is most obvious when a standardization of individual smoothed estimates leads to correlation estimations outside the interval  $[-1, 1]$ , or when portfolios with variances lower zero become possible.

$\hat{\rho}_{(\cdot)k,l}(t) = \frac{\hat{\sigma}_{(\cdot)k,l}^2(t)}{\hat{\sigma}_{(\cdot)k}(t)\hat{\sigma}_{(\cdot)l}(t)}$  to nonparametric correlations exhibit (mostly) a smooth but heteroscedastic course over time. Statistical tests on the differences of correlation matrices in two dates confirm the time-variation. Since bandwidths of elements differ without a functional relationship, the shortcomings of undefined estimates  $\hat{\rho}_{(\cdot)k,l}(t) \notin [-1, 1]$  and not positive semidefinite matrices are observed.<sup>36</sup> Turning to the wholistic smoothing conception we gain consistent two- and one-sided covariance matrix estimates  $\hat{\Sigma}^2(t)$  and  $\hat{\Sigma}_{(1)}^2(t)$  based on global bandwidths  $\bar{h}^{\text{opt}}$  or  $\bar{h}_{(1)}^{\text{opt}}$ , respectively. The above standardization to correlations is now coherent and the  $30 \times 30$  matrix estimates vary still significantly over time. With a successful (component-wise) fit of the Pearson VII density  $f_{\varepsilon_k; m+, c+, m-, c-}^{\text{VII}}$  to realized innovations (apart from a few normal approximations of weaker tails) the complete non-stationary modeling works for the extensive empirical setup. In a forecasting experiment, the non-stationary approach is calibrated again on the in-sample part from 1999 to 2006 and distributional forecasts from the one-sided, wholistic setup are assessed in the out-of-sample years 2007 to 2008. By dint of the fitted return dynamics  $\mathbf{X}_t$  we predict the distribution of the 1-day ahead return vector  $\mathbf{X}_{t+1}$  and its portfolio return  $W_{t+1}$  for exemplary portfolios with a 30-dimensional exposure vector according to the risk factors. The modeling performance is evaluated with the Kupiec test on shortfall rates against forecasted return quantiles  $\text{VaR}_{W, \text{RM}}^{1-\alpha, 1d}(t)$ , that are derived both analytically and with a Monte-Carlo simulation (using random residuals and Cholesky decomposition of correlation matrices). We compare the forecasting ability to a delta-normal model, where our regression approach clearly outperforms the basic risk model. Altogether, the return forecasts of the non-stationary model are satisfactory, the hypothesis is not accepted at all confidence levels but its analytical and simulative VaR forecasts cause shortfall numbers close to the boundaries of the Kupiec test. It replicates multivariate return dynamics well and our empirical studies provide evidence for a flexible and manageable approach.

Besides the successful aggregate test of the model approximation, we do not explicitly verify all singular assumptions of the non-stationary return model. For that task we refer back to Herzel et al. (2005), who elaborately proved on a tri-variate example (exchange rate EUR/USD, FTSE 100 index, 10-year US T-bond rate) that estimated innovation vectors are iid in time and have independent coordinates, added by goodness-of-fit tests of the Pearson VII. In their empirical studies 1-day ahead multivariate return forecasts and 10- and 20-day univariate density forecasts support the non-stationary paradigm and an outperformance against the RiskMetrics<sup>TM</sup> (JP Morgan) approach is proved. The careful modeling of extremal behavior of innovations was identified as one factor of success. Gürtler and Rauh (2009) provide empirical evidence that the nonparametric regression model approximates daily return series of several financial instruments (identical benchmarks as we analyze in sections 2 to 4) univariate well. It outperforms parametric VaR implementations as they test model assumptions and evaluate shortfall rates. Extended studies of Gürtler and Rauh (2012) compare the volatility structures and the whole univariate adaptations with ARCH-type models, resulting in modeling dominances of the non-stationary approach.

Following the abilities and the practicability of the non-stationary, nonparametric regression model, we propose several fields for future research: As optimal parameters (bandwidths and Pearson VII coefficients) are observed to change through time, a time-varying setup that is reestimated with a certain frequency is worth to be considered. A full nonparametric setup is conceivable, that substitutes the (still restrictive) para-

<sup>36</sup>The contradiction to definiteness  $\bar{h}$  could also arise if all matrix elements are well-defined.

metric approach for the innovation's distribution by a nonparametric kernel density. Going along with the ideas of Mikosch and Starica (2003) the inclusion of a time-dependent expected yield, modeled with kernel regression, may be a further step. Under the basic belief that exogenous economic factors and resulting return dynamics evolve smoothly through time, the framework may be used for portfolio optimization in terms of a tactical asset allocation.

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