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Optimal Cartel Pricing in the Presence of an Antitrust Authority*

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Abstract

The dynamic behavior of a price-fixing cartel is explored when it is concerned about creating suspicions that a cartel has formed. The intertemporal structure of the price path is characterized and the effect of antitrust policy on the cartel's steady-state price is explored.

*I want to thank Bates, White and Ballantine for re-stimulating my interest in this topic. I'd also like to acknowledge the comments of Myong Chang, with whom I originally discussed this topic more than ten years ago, Jimmy Chan, Fred Chen, Massimo Motta, participants of presentations at Wake Forest, Toronto, Penn, Dept. of Justice, George Mason, Hopkins, and EARIE 2001, and the enthusiastic research assistance of Joe Chen. This research is supported by the National Science Foundation.

1 Introduction

> Since the beginning of FY 1997, the Antitrust Division has prosecuted international cartels affecting over $10 billion in U.S. commerce ... [These cartels] have been bigger, in terms of the volume of affected commerce and the amount of harm caused to American businesses and consumers, than any conspiracies previously encountered by the Antitrust Division. [*Annual Report, Antitrust Division, United States Department of Justice*, 1999: pp. 5-6]

> International cartels are estimated to represent a drain of hundreds of millions of euros on the European economy. ... Since 1998, the number of cartel cases investigated by the Commission has increased dramatically. [*European Community Competition Policy, XXXth Report on Competition Policy*, 2000: pp. 24-25]

As these quotes from American and European antitrust authorities suggest, price-fixing remains a perennial problem which makes it all the more important that we understand when cartels form and, when they do form, how they behave. Though there is a voluminous theoretical literature on collusive pricing, an important dimension to price-fixing cartels has received little attention. In light of the illegality of price-fixing**, a critical goal faced by a cartel is to avoid the appearance that there is a cartel.** Firms want to raise prices but not suspicions that they are coordinating their behavior. If high prices or rapidly increasing prices or, more generally, anomalous price movements may make customers and the antitrust authorities suspicious that a cartel is operating, one would expect this to have implications for how the cartel prices.

This paper is the initial step in a research project whose objective is to explore cartel pricing in the presence of detection considerations. Some of the questions to be addressed include: What are the intertemporal properties of the collusive price path? How does the decision to form a cartel and the properties of the collusive price path respond to various instruments of antitrust policy? What types of industry traits make detection more difficult and what are the implications of those traits for cartel pricing?

Towards beginning to address these questions, this paper makes two contributions. First, it characterizes the intertemporal structure to the joint profit maximizing price path when two dynamics are at work - higher prices increase penalties in the event of detection and bigger price changes make detection more likely. In spite of the potential complexity of these dynamics, the cartel price path is shown to be monotonically increasing under

general assumptions. The cartel gradually raises price as it balances off increasing profit with increasing the probability of detection. The second contribution is to explore how antitrust policy impacts the steady-state cartel price. While some results confirm existing intuition about the impact of antitrust policy, some yield a new intuition. Comparative statics on the steady-state price reveal that it is decreasing in the damage multiple and the probability of detection; both of which confirm existing intuition. However, it is independent of the level of fixed fines. Furthermore, if fines are the only penalty, the cartel's steady-state price is the same as in the absence of antitrust laws. The equivalence between fines and damages found in previous work is then shown to break down in the context of a dynamic model. A second surprising result is that a more stringent standard for calculating damages actually increases the cartel's steady-state price. Finally, this model of detection is augmented by allowing both higher prices as well as bigger price changes to make detection more likely. Numerical analysis reveals a unique and potentially identifying pricing pattern - the cartel gradually raises price but then price moderately declines to its steady-state value.

Related Work A few papers have investigated, in a static setting, optimal cartel pricing under the constraint of possible detection. Block, Nold, and Sidak (1981) consider a static oligopoly model in which the probability of detection depends on the price-cost margin and the penalty is a multiple of above-normal profits. They show that the optimal cartel price is below the monopoly price and that the cartel price is decreasing in the penalty multiple and the level of enforcement expenditures (higher levels of which raise the probability of detection). Spiller (1986), Salant (1987), and Baker (1988) extend the static formulation to allow buyers to adjust their purchases under the anticipation that they may be able to collect multiple damages if sellers are shown to have been colluding. Also within a static setting, Besanko and Spulber (1989, 1990), LaCasse (1995), Polo (1997), and Souam (2001) explore a context in which firms have private information, which influences whether or not they collude, and either the government or buyers must decide whether to pursue costly legal action. Three papers consider a dynamic setting. Cyrenne (1999) modifies Green and Porter (1984) by assuming that a price war, and the ensuing raising of price after the war, results in detection for sure and with it a fixed fine. Spagnolo (2000) and Motta and Polo (2001) consider the effects of leniency programs on the incentives to collude when the probability of detection and penalties are both fixed. Though considering collusive behavior in a dynamic setting with antitrust laws, these papers exclude the sources of dynamics that are the foci of the current analysis;

specifically, that the probability of detection and penalties are sensitive to firms' pricing behavior. It is that sensitivity that will generate predictions about cartel pricing dynamics.

2 Model

The representative firm's profit when all firms charge a price of $P \in \Omega$ is denoted $\pi(P)$ where Ω is the set of feasible prices. If market demand is $D(\cdot)$ and a firm's cost function is $C(\cdot)$ then the profit function is $\pi(P) = P(D(P)/n) - C(D(P)/n)$, given $n \geq 2$ firms. In the absence of the formation of a cartel, a symmetric equilibrium is assumed to exist which entails a price of $\widehat{P}$ and firm profit of $\widehat{\pi} \geq 0$.

If firms form a cartel, they meet to determine price. Assume these meetings, and any associated documentation, provides the "smoking gun" if an investigation is pursued.[1] The cartel is detected with some probability and incurs penalties in that event. Detection can be viewed as the end of the horizon with a terminal payoff of $[\widehat{\pi}/(1-\delta)] - X^t - F$ where X^t is a firm's damages in the event the cartel is detected, F is any (fixed) fines, and $\delta \in (0,1)$ is the discount factor.[2] In this model, damages refers to any penalty that is sensitive to the prices charged while fines refer to penalties that are fixed with respect to the endogenous variables.[3] If not detected, collusion continues on to the next period. There is an infinite number of periods. Penalties are assumed to be sufficiently bounded from above for all histories so that the expected present value of a firm's income stream is always positive and thus bankruptcy is avoided.

A cartel member's damages, denoted X^t for period t, are assumed to evolve in the following manner:

$$X^t = \beta X^{t-1} + \gamma x\left(P^t\right) \text{ where } \beta \in [0,1), \gamma \geq 0, \tag{1}$$

[1] Though it is assumed that an investigation leads to conviction with probability one, all results would go through if the probability of conviction is only required to be positive.

[2] One could allow for the cartel to be reestablished sometime in the future and I suspect many results would not change. Of the 1300 firms indicted by the Department of Justice over 1962-1980, 14% were recidivists (Bosch and Eckard, 1991).

[3] Though this use of the term "fines" is standard in the literature, in recent years U.S. Department of Justice fines have become sensitive to the length of the cartel and the prices charged. Federal Sentencing Guidelines provide for fines equal to 20% of the value of affected commerce multiplied by a culpability score which lies between 2 and 4 (American Antitrust Institute, 12/7/01). However, the sensitivity of actual penalties at the margin is not so clear. For example, according to these guidelines, Hoffman-La Roche should have been levied a penalty between $1.3 and 2.6 billion and it was instead required to pay $500 million due to some final adjustment. Also, prison sentences are probably quite insensitive to marginal changes in prices so that their monetary valuation would make up part of F.

where P^t is the cartel price. As time progresses, damages incurred in previous periods become increasingly difficult to document and $1-\beta$ measures the rate of the deterioration of the evidence.[4] $x\left(P^t\right)$ is the level of damages incurred in the current period where γ is the multiple of damages that a firm can expect to pay if found caught colluding. While U.S. antitrust law specifies treble damages, γ could be less than three because a case is settled out-of-court. Single damages are not unusual for an out-of-court settlement.[5] Current U.S. antitrust practice is $x\left(P^t\right)=\left(P^t-\widehat{P}\right)\left(D\left(P^t\right)/n\right)$.[6]

Detection of a cartel can occur from many sources; some of which are related to price - such as customer complaints - and some of which are unrelated to price - such as internal whistleblowers.[7] Hay and Kelley (1974) find that detection was attributed to a complaint by a customer or a local, state, or federal agency in 13 of 49 price-fixing cases. In the recent graphite electrodes case, it was reported that the investigation began with a complaint from a steel manufacturer which is a purchaser of graphite electrodes (Levenstein and Suslow, 2001). Anomalous pricing may cause customers to become suspicious and pursue legal action or share their suspicions with the antitrust authorities.[8] Though it isn't important for my model, I do imagine that buyers (in many price-fixing cases, they are industrial buyers) are the ones who are becoming suspicious about collusion.

> As a general rule, the [Antitrust] Division follows leads generated by disgruntled employees, unhappy customers, or witnesses from ongoing investigations.

[4] Assuming a depreciation rate to damages is important analytically as it bounds the penalty. An alternative approach is to impose a statute of limitations so that the damage penalty is the sum of damages incurred over a bounded number of periods into the past. I conjecture the same type of insight would emerge under such an assumption. I thank Ted O'Donoghue for making this suggestion. β can also capture the fact that the real value of the damages declines over time as defendants are not required to pay foregone interest; interest is applied only after the judicial determination of an antitrust violation. Blackstone and Bowman (1987) estimate that this reduced the real value of damage penalties by around 50% in 1975 given the average length of a cartel around that time was 8.6 years.

[5] See Connor (2001) and White (2001) for some estimates of damages associated with the lysine cartel. Also see de Roos (1999) for an analysis of the lysine cartel.

[6] "[After the] court selects a 'competitive price,' [it] ... awards the plaintiff the difference between the competitive estimate and the amount paid, times the quantity purchased, trebled." (Breit and Elzinga, 1986, p. 21.)

[7] Bryant and Eckard (1991) estimate the chances of a price-fixing cartel being indicted in a 12-month period to be around 15%.

[8] The Nasdaq case is one in which truly anomalous pricing resulted in suspicions about collusion. It was scholars rather than market participants who observed that dealers avoided odd-eighth quotes and ultimately explained it as a form of collusive behavior (Christie and Schultz, 1994). Though the market-makers did not admit guilt, they did pay an out-of-court settlement of around $1 billion.

> As such, it is very much a reactive agency with respect to the search for criminal antitrust violations. ... Customers, especially federal, state, and local procurement agencies, play a role in identifying suspicious pricing, bid, or shipment patterns. [McAnney, 1991, pp. 529, 530]

In modelling the detection process, there isn't much relevant evidence to offer guidance and it is not well-understood how people identify anomalous events. I have then decided to take a more agnostic approach by specifying a class of probability of detection functions and exploring how properties of those functions influence cartel pricing dynamics. Letting $\phi\left(P^t, P^{t-1}\right)$ denote the probability of detection in period t, it is allowed to depend on the current price and the previous period's price. One can interpret $\phi\left(\widehat{P}, \widehat{P}\right)$ as a baseline probability of detection driven by, for example, internal whistleblowers. The inclusion of a more comprehensive price history would significantly complicate the analysis - greatly expanding the state space - without any apparent gain in insight. Allowing just the most recent past to matter is potentially significant, however, as price changes can then play a role in detection.

This modelling of detection warrants some further discussion since it does not explicitly model those agents who might engage in detection. The first point to make concerns tractability. Even with a single agent (that is, the cartel), this is a complex model with two state variables, $\left(P^{t-1}, X^{t-1}\right)$, and thereby two distinct sources of dynamics - detection and antitrust penalties. As currently formulated, the model is rich enough to provide new insight into cartel pricing dynamics, even with a simple modelling of the detection process, and a more complex model at this stage is likely to prove intractable. Tractability issues aside, there is another motivation for this approach. The objective of this paper is not to develop insight and testable hypotheses about detection but rather about cartel pricing. A good model of the detection process is then defined to be one that is a plausible description of how cartel members *perceive* the detection process. To my knowledge, there is little evidence from past cases that cartels hold a sophisticated view of buyers (which is implied if one were to model buyers as strategic agents and derive an equilibrium). It strikes me as quite reasonable that firms might simply postulate that higher prices or bigger price changes result in a greater likelihood of creating suspicions without having derived that relationship from first principles about buyers. Thus, even if this modelling of the detection process is wrong, the resulting statements about cartel pricing may be accurate if that model is a reasonable representation of firms' perceptions.[9]

[9] Nor do I believe it is inconsistent to model firms as choosing prices optimally - as that is a statement

In period 1, firms have the choice of forming a cartel, and risking detection and penalties, or earning non-collusive profit of $\widehat{\pi}$. If they choose the former, they can, at any time, choose to discontinue colluding. In that event, it is assumed they'll never collude again and receive a terminal payoff of $[\widehat{\pi}/(1-\delta)] - \sigma\left(P^{t-1}, X^{t-1}\right)$ where the last period of collusion is period $t-1$. $\sigma\left(P^{t-1}, X^{t-1}\right)$ is to be interpreted as the present value of the expected penalty when collusion is discovered after the dissolution of the cartel (for example, incidental discovery of incriminating documents in an unrelated legal case).

For the purposes of establishing the existence of an optimal cartel price path, the following assumptions are imposed. Additional structure will be required to derive properties of that path.[10]

A1 $\pi : \Omega \to \Re$ is bounded and continuously differentiable and $\exists P^m > \widehat{P}$ such that $\pi'(P) \gtreqless 0$ as $P \lesseqgtr P^m$.

A2 $x : \Omega \to \Re_+$ is bounded, continuously differentiable, and non-decreasing.

A3 $\phi : \Omega^2 \to [0,1]$ is continuous.

A4 $\sigma : \Omega \times \Re_+ \to \Re_+$ is bounded, continuous, and non-decreasing.

A5 Ω is a compact convex subset of $\Re_+$ and $\left[\widehat{P}, P^m\right] \subseteq \Omega$.

The cartel chooses an infinite price path so as to maximize the expected sum of discounted income. To break indifference, firms are assumed to collude if they are indifferent between colluding and not colluding. It is important to emphasize that we do not ignore the usual incentive compatibility constraints which ensure that a firm will go along with the collusive price path. One can cast the preceding model as an infinite-horizon perfect monitoring (though non-repeated) game played among the n firms. The joint profit-maximizing price path that is characterized here is then the best symmetric equilibrium price path when δ is sufficiently close to one; that is, when these incentive compatibility constraints do not bind. Given the complexity of the dynamics associated with detection and antitrust penalties, it makes sense to initially characterize this price path which, as the reader will see, is a substantive task in itself.

about what one thinks is best for one's self - and, at the same time, suppose that firms do not derive buyers' optimal detection behavior - as that is a statement about what one thinks is best for others. An agent may know what is best for themselves without having a clue as to what is best for someone else.

[10] If $x(P) = \left(P - \widehat{P}\right)(D(P)/n)$ then it could be decreasing for sufficiently high prices which contradicts A2. However, it is shown in Harrington (2001) that, under standard assumptions on demand and cost functions, x is increasing for prices on the optimal price path.

3 Existence of an Optimal Price Path

The basic problem is one of the cartel manager choosing a price path to maximize the expected present value of the representative cartel member's income stream. To establish the existence of an optimal price path, dynamic programming is used. The state variables are yesterday's price, P^{t-1}, and accumulated damages, X^{t-1}. $V\left(P^{t-1}, X^{t-1}\right)$ denotes the value function when the cartel is still functioning as of period t and is defined as the fixed point to:

$$\begin{aligned} V\left(P^{t-1}, X^{t-1}\right) = \quad & \max_{P\in\Omega} \pi\left(P\right) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta X^{t-1} - \gamma x\left(P\right) - F\right] \\ & +\delta\left[1-\phi\left(P, P^{t-1}\right)\right]\max\left\{V\left(P, \beta X^{t-1} + \gamma x\left(P\right)\right), \left(\widehat{\pi}/\left(1-\delta\right)\right) - \sigma\left(P, \beta X^{t-1} + \gamma x\left(P\right)\right)\right\}. \end{aligned} \tag{2}$$

$\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta X^{t-1} - \gamma x\left(P\right) - F$ is the terminal payoff associated with the cartel being detected. Also note that firms have the future option of dismantling the cartel and receiving a terminal payoff of $\left(\widehat{\pi}/\left(1-\delta\right)\right) - \sigma\left(P, \beta X^{t-1} + \gamma x\left(P\right)\right)$. All proofs are in Appendix A.

Theorem 1 *Assume A1-A5. An optimal price path exists.*

A natural specification for the post-cartel penalty function is

$$\sigma\left(P^{t-1}, X^{t-1}\right) = \sum_{\tau=t}^{\infty} \delta^{\tau-t+1}\left[\beta^{\tau-t+1} X^{t-1} + F\right]\omega^{\tau}\left(P^{t-1}\right) \tag{3}$$

where $\omega^{\tau}\left(P^{t-1}\right)$ is the probability of the cartel being discovered in period τ (which may depend on the initial conditions for price for the post-cartel period). In that case, $\sigma\left(P^{t-1}, X^{t-1}\right)$ is an affine function of X^{t-1}. This property is used in the next result which shows that the value function is a decreasing convex function of accumulated damages.

Theorem 2 *Assume A1-A5 and σ is a weakly concave function of X^{t-1}. $V\left(P^{t-1}, X^{t-1}\right)$ is a decreasing convex function of X^{t-1}.*

As a higher value for X^{t-1} means a more severe penalty in the event of detection, it is unsurprising that the value of collusion is decreasing in the amount of accumulated damages. It is also easy to explain why the value function is convex. Holding the price path fixed and assuming collusion is infinitely-lived, a firm's period t payoff is linear and decreasing in X^{t-1} as the expected present value of the penalty associated with X^{t-1} is

$X^{t-1}\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t+1}\omega^{\tau}$ where ω^{τ} is the probability of detection in period τ.[11] Since the cartel can partially mitigate the effect of increased accumulated damages by adjusting the price path to make detection less likely, the value function, at each value of X^{t-1}, is bounded from below by a linear decreasing function of X^{t-1} which is tangent to the value function at that value of X^{t-1}. With this lower bound, the value function is then (weakly) convex in X^{t-1}.

4 Properties of the Price Path

Given existence, the next step is to characterize the intertemporal structure of the price path which will lay the foundation for then exploring how antitrust policy impacts cartel pricing. Obviously, additional structure on the probability of detection function is required to yield a useful characterization of cartel pricing. Previous static analyses of the influence of antitrust law on cartel pricing assume the probability of detection depends only on the price level and is an increasing function (for example, Block et al, 1981). I initially explored this case but found nonsensical results; after raising price in the first period, the cartel steadily lowers price (Harrington, 2001). The intuition is quite general. As firms collude over time, one can show that accumulated damages on an optimal cartel price path grow which means a higher penalty in the event of detection. Since the probability of detection is increasing in price, a natural response to a higher potential penalty is to lower price and thereby reduce the likelihood of detection. Thus, firms steadily lower price over time so as to make detection less likely. To my knowledge, there is no empirical evidence for such a cartel price path. Indeed, it is quite contrary to the steadily rising price paths documented in the citric acid cartel of 1987-97 (Connor, 1998) and the graphite electrodes cartel of 1992-97 (Levenstein and Suslow, 2001). In that a falling price path is the logical implication of having detection depend only on the price level, I infer that detection is not largely driven by the price level. A natural alternative is that detection is driven instead by price changes. That is the avenue I will pursue here. However, I will later explore allowing detection to depend on both price changes and price levels.

In specifying properties for the probability of detection function, the basic story to have in mind is that the environment is perceived to be stable so that cartel members expect buyers to anticipate price being fairly stable. Thus, bigger price changes - up and even possibly down - are more likely to be perceived as anomalous and thus trigger

[11] If collusion is finitely-lived then one has the same type of expression up until the final period of collusion and then σ is relevant thereafter.

suspicions about the presence of a cartel.

A6 $\exists\widehat{\phi}:\Re\rightarrow[0,1]$ and $g:\Omega\rightarrow\Re_{++}$, where g is a strictly positive, non-increasing, continuously differentiable function, such that

$$\phi\left(P^{t},P^{t-1}\right)=\widehat{\phi}\left(\left(P^{t}-P^{t-1}\right)g\left(P^{t-1}\right)\right)\ \forall\left(P^{t},P^{t-1}\right)\in\Omega^{2}.$$

A7 If $x\geq y\geq0$ then $\widehat{\phi}\left(x\right)\geq\widehat{\phi}\left(y\right)$.

A8 $\widehat{\phi}\left(x\right)\geq\widehat{\phi}\left(0\right)\ \forall x\in\Re$ and $\widehat{\phi}\left(0\right)\in[0,1)$.

A9 $\exists\varepsilon>0$ such that $\widehat{\phi}$ is continuously differentiable in an ε-ball around $\forall P''\in P',\forall P'\in\Omega$ and $\widehat{\phi}'\left(0\right)=0$.

A6-A8 specify that the probability of detection depends on the change in price, is non-decreasing for price increases, and is minimized by keeping price constant. Note that if g is a constant then the probability of detection depends only on the size of price movements while if $g\left(P^{t-1}\right)=1/P^{t-1}$ then it depends on the percentage change in price. A9 requires differentiability around a price change of zero and is a necessary technical condition.[12] Though we state $\widehat{\phi}'\left(0\right)=0$ as an assumption, it actually follows from A8 and assuming the derivative of $\widehat{\phi}$ at a price change of zero is defined. Two additional assumptions involving the profit function are required.

A10 $\pi\left(P\right)-\delta\widehat{\phi}\left(0\right)\left[\left(\frac{\gamma x(P)}{1-\beta}\right)+F\right]>\widehat{\pi}\ \forall P\in\left(\widehat{P},P^{m}\right]$.

A11 $\exists P^{*}\in\left(\widehat{P},P^{m}\right]$ such that

$$\pi'\left(P\right)-\left[\delta\widehat{\phi}\left(0\right)/\left(1-\delta\beta\left(1-\widehat{\phi}\left(0\right)\right)\right)\right]\gamma x'\left(P\right)\gtreqless0\text{ as }P\lesseqgtr P^{*}.$$

In Harrington (2001), it is shown that A10 is sufficient to ensure that, at a steady state price of P, colluding is preferable to not colluding. A11 requires quasi-concavity of an income function which is defined to be profit less some multiple of damages. It is shown later that these assumptions are satisfied under standard conditions on demand and cost functions.

[12] I want to acknowledge Ali Khan for the proper statement of A9. He developed an elegant example which showed that a function can be differentiable at a point but not be differentiable in an ε-ball around that point.

4.1 Monotonicity of the Price Path

Theorem 3 shows that collusion is infinitely-lived, involves a non-decreasing price path, and the long-run price is P^* (as defined in A11).[13] These properties for the price path are derived when firms choose to cartelize.[14]

Theorem 3 *Assume A1-A11 and* $P^0 \in \left[\widehat{P}, P^*\right)$. *If it is optimal to form a cartel then it is optimal to collude in all periods and if* $\{P^t\}_{t=1}^{\infty}$ *is an optimal price path then: i) it is non-decreasing over time; and ii)* $P^t \to P^*$ *as* $t \to \infty$.

In spite of the generality of the structure, the price path is well-behaved in being monotonic. The intuition is immediate. In that larger price movements result in a higher probability of detection, the optimal price path has the cartel gradually increase price to its long-run target value of P^* with the hope of not triggering suspicions. A numerical example in Figure 1 shows a typical price path when the probability of detection function is strictly increasing in price increases. Price starts at the non-collusive (Cournot) price of 333 and is gradually raised; asymptoting a value of 470 which is below the simple monopoly price of 500.[15] Let me note that if the probability of detection is fixed at $\widehat{\phi}(0)$ - so that it is independent of price - then the cartel immediately increases price to P^* and leaves it there. Intuitively, when the probability of detection is fixed then the expected penalty associated with past damages is independent of what firms do (as long as they continue colluding). Those damages are sunk. Hence, the optimal price doesn't change over time even though damages do grow.

Though the equations characterizing the dynamic path of price are rather complex, there is a simple equation defining the long-run cartel price, P^*, which makes it conducive for performing comparative statics. P^* is defined as the unique solution to

$$\pi'(P^*) - \left[\delta\widehat{\phi}(0) / \left(1 - \delta\beta\left(1 - \widehat{\phi}(0)\right)\right)\right] \gamma x'(P^*) = 0. \tag{4}$$

[13] Without A11, the proof of Theorem 3 still establishes that the price path is non-decreasing and is bounded from above by P^*. A11 serves to show that $lim_{t\to\infty} P^t = P^*$.

[14] Here are two sets of sufficient conditions for cartel formation to occur when $P^0 = \widehat{P}$ and $X^0 = 0$. First, γ and F are sufficiently small. Second, $x\left(\widehat{P}\right) = 0$ and $F = 0$. The first case is immediate and the second case is shown in Harrington (2001). The latter is robust to small changes in the assumptions.

[15] The numerical analysis assumes market demand of $1000 - P^t$, constant marginal cost of zero, $\widehat{P}$ is the Cournot price, the damage function is $x\left(P^t\right) = \left(P^t - \widehat{P}\right)\left(D\left(P^t\right)/n\right)$, and $\sigma = 0 \forall \left(P^{t-1}, X^{t-1}\right)$. Parameters are $n = 2$, $\beta = .6$, $\gamma = 1$, $\delta = .96$, and $F = 0$. The probability of detection function is $\phi\left(P^t, P^{t-1}\right) = \min\left\{.05 + .0002592\left(P^t - P^{t-1}\right)^2, 1\right\}$ so that raising the price 25% of the way from the non-collusive to the simple monopoly price in one period results in a 50% chance of detection. More numerical results are in Harrington (2001).

The long-run cartel price then depends on the damage function and multiple, the rate at which damages depreciate, and the probability of detection function. If the profit function is concave ($\pi'' < 0$), the damage function is strictly increasing ($\gamma x' > 0$), and the minimum probability of detection is positive ($\widehat{\phi}(0) > 0$) then $P^* < P^m$ so that the cartel price is bounded below the simple monopoly price in all periods. Thus, antitrust law constrains pricing behavior. However, note that if $\gamma = 0$, so that the only penalty is fixed fines, then it follows from (4) that $P^* = P^m$. *At the steady-state, fixed fines do not constrain the cartel's price.* It is true, however, that higher fines can be expected to affect the speed with which price is raised and, if fines are sufficiently high, they can deter cartel formation altogether. This is summarized as Remark 1.

Result 1 The steady state cartel price is less than the simple monopoly price when penalties include damages. The steady-state cartel price equals the simple monopoly price when the only penalty is fixed fines (assuming cartel formation occurs).

This independence result with respect to fines can be explained as follows. In the long run, price settles down so that price changes converge to zero. Given that $\widehat{\phi}'(0) = 0$, marginal changes in price have no first-order effect on the probability of detection though continue to have a first-order effect on the potential penalty through the damage function. Thus, factors that influence the relationship between price and the size of the penalty - the discount factor, the rate of depreciation of damages, the damage multiple, and the damage function - all influence the long-run price. As a result, if there are only fines and no damages then, as price changes go to zero, marginal changes in price have no effect on the expected penalty so that the cartel price converges to the simple monopoly price. Recall that $\widehat{\phi}'(0) = 0$ follows from differentiability of the probability of detection function and that the probability of detection is minimized at a price change of zero; both assumptions being quite reasonable when the environment is stationary. Thus, a fairly general implication is that fixed penalties have no long-run effect on the cartel price.

The independence of the steady-state cartel price with respect to fixed penalties is in stark contrast to static models of collusive pricing in the presence of antitrust laws. In those models, there is an equivalence between fines and damages in the sense that any price resulting for some damage multiple could alternatively be generated through an appropriately selected fine.[16] In contrast, when detection depends on price changes in a dynamic model, price is bounded below the simple monopoly price when penalties

[16] To see this point, consider a static model in which the cartel maximizes profit less expected penalties and let $\psi(P)$ denote the probability of detection (note that it only depends on the price level). When

include damages but instead converges to the simple monopoly price when damages are not deployed. Thus, if antitrust policy is intended to constrain cartel prices, even in the long run, it is essential that penalties be responsive to the price charged.

Finally, the steady-state price can also be independent of the damage multiple though it requires that damages are proportional to profit. If $x\left(P^t\right) = \theta\pi\left(P^t\right)$ for some $\theta > 0$ then (4) once again implies $P^* = P^m$. For example, this proportionality occurs under the standard damage formula of $x\left(P^t\right) = \left(P^t - \widehat{P}\right)\left(D\left(P^t\right)/n\right)$ when marginal cost is constant and the but-for price is the competitive price.

4.2 Comparative Statics

Assume the market demand function, $D(\cdot)$, is twice differentiable and each firm has constant marginal cost of c. A firm's profit is then $\pi(P) = (P-c)\left(D(P)/n\right)$. Further assume $D''(P) \leq 0$ so that A1 holds. Next suppose that the damage function is $x(P) = \left(P - \widehat{P}\right)\left(D(P)/n\right)$ where $\widehat{P} > c$. To ensure that A11 is satisfied, define

$$\Psi(P) \equiv \pi(P) - \kappa d(P) = (1/n)\left[(P-c)D(P) - \kappa\left(P - \widehat{P}\right)D(P)\right]$$

$$\text{where } \kappa \equiv \delta\widehat{\phi}(0)\gamma \Big/ \left(1 - \delta\beta\left(1 - \widehat{\phi}(0)\right)\right).$$

Note that if $\Psi''(P) < 0$ then P^* is defined by $\Psi'(P^*) = 0$. Taking the first two derivatives of Ψ:

$$\begin{aligned}
\Psi'(P) &= (1/n)\left\{(1-\kappa)\left[(P-c)D'(P) + D(P)\right] + \kappa\left(\widehat{P} - c\right)D'(P)\right\}, \qquad (5)\\
\Psi''(P) &= (1/n)\left\{(1-\kappa)\left[2D'(P) + (P-c)D''(P)\right] + \kappa\left(\widehat{P} - c\right)D''(P)\right\}.
\end{aligned}$$

$\Psi''(P) < 0$ if $\kappa < 1$ and $D'' \leq 0$. For P^* to exceed $\widehat{P}$, one needs:

$$\Psi'\left(\widehat{P}\right) = (1/n)\left\{(\widehat{P} - c)D'\left(\widehat{P}\right) + (1-\kappa)D\left(\widehat{P}\right)\right\} > 0. \qquad (6)$$

Since $(\widehat{P} - c)D'\left(\widehat{P}\right) + D\left(\widehat{P}\right) > 0$, as $\widehat{P}$ is associated with the non-collusive outcome, then $\Psi'\left(\widehat{P}\right) > 0$ if κ is sufficiently close to zero which holds, for example, if either the

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the penalty is damages, the expected penalty is $\psi(P)\gamma x(P)$ and when the penalty is fines, the expected penalty is $\psi(P)F$. The optimal cartel price is defined by that price which equates marginal profit with marginal expected penalty. Next suppose a price of $\overline{P}$ is induced by a policy of damages:

$$\pi'\left(\overline{P}\right) = \psi'\left(\overline{P}\right)\gamma x\left(\overline{P}\right) + \psi\left(\overline{P}\right)\gamma x'\left(\overline{P}\right).$$

We can then induce that same price with fines by setting F so that

$$\psi'\left(\overline{P}\right)F = \psi'\left(\overline{P}\right)\gamma x\left(\overline{P}\right) + \psi\left(\overline{P}\right)\gamma x'\left(\overline{P}\right) \Leftrightarrow F = \gamma x\left(\overline{P}\right) + \left[\psi\left(\overline{P}\right)/\psi'\left(\overline{P}\right)\right]\gamma x'\left(\overline{P}\right).$$

Thus, any price can be implemented either by fines or damages.

probability of detection or the damage multiple is sufficiently small. P^* is then defined by:

$$(1-\kappa)\left[(P^*-c)D'\left(P^*\right)+D\left(P^*\right)\right]+\kappa\left(\widehat{P}-c\right)D'\left(P^*\right)=0. \tag{7}$$

Taking the total derivative of (7) with respect to κ,

$$\frac{\partial P^*}{\partial \kappa}=\frac{\left[(P^*-c)D'\left(P^*\right)+D\left(P^*\right)\right]-\left(\widehat{P}-c\right)D'\left(P^*\right)}{(1-\kappa)\left[2D'\left(P^*\right)+\left(P^*-c\right)D''\left(P^*\right)\right]+\kappa\left(\widehat{P}-c\right)D''\left(P^*\right)}<0. \tag{8}$$

It is straightforward to show that κ is increasing in γ, $\widehat{\phi}\left(0\right)$, β, and δ. The following intuitive results are then immediate.

Result 2 The steady-state cartel price is reduced when: i) the damage multiple, γ, is increased; ii) the probability of detection, $\widehat{\phi}\left(\cdot\right)$, is increased; iii) the rate at which damages persist over time, β, is increased; and iv) the discount factor, δ, is increased.

Numerical analysis reveals that when a change in a parameter causes the long-run cartel price to fall (rise), the entire price path declines (rises); see Harrington (2001). The first three results are quite immediate. To explain the last one, note that the cartel faces an intertemporal trade-off in that a higher price in the current period raises current profit but lowers the future payoff by increasing the likelihood of detection and, in the event of future detection, increasing the penalty. As cartel members become more patient, they then prefer lower cartel prices.

A final interesting comparative static exercise is to consider the influence of the but-for price, $\widehat{P}$, on the steady-state cartel price. Recall that the but-for price is the price used in calculating damages.[17] It will be useful to generalize the damage function to:

$$x\left(P\right)=\left(P-\widehat{P}\right)\left[\alpha\left(D\left(P\right)/n\right)+\left(1-\alpha\right)\left(D\left(\widehat{P}\right)/n\right)\right] \tag{9}$$

where $\alpha\in\left[.5,1\right]$. U.S. antitrust practice is captured by $\alpha=1$ while if damages were specified to equal the loss in consumer surplus then $\alpha=.5$, using a linear approximation. Note that as α falls, the cartel's price has less of an influence on the level of demand used for calculating damages. It is straightforward to derive:

$$\frac{\partial P^*}{\partial \widehat{P}}=\frac{\kappa\left[\left(1-\alpha\right)D'\left(\widehat{P}\right)-\alpha D'\left(P^*\right)\right]}{(1-\kappa\alpha)\left[2D'\left(P^*\right)+\left(P^*-c\right)D''\left(P^*\right)\right]+\kappa\alpha\left(\widehat{P}-c\right)D''\left(P^*\right)}. \tag{10}$$

[17] Actually, $\widehat{P}$ represents two different prices: the non-collusive price and the but-for price. While, in practice, they are intended to be the same, in principle they could be different. The point to make is that it is $\widehat{P}$ as the but-for price which influences the steady-state cartel price.

As before, the denominator is negative. Since the numerator is increasing in α, it is minimized at $\alpha = .5$ and, therefore, the numerator is non-negative as long as $D'\left(\widehat{P}\right) \geq D'\left(P^*\right)$. Since $P^* > \widehat{P}$ then $D'' \leq 0$ implies $D'\left(\widehat{P}\right) \geq D'\left(P^*\right)$. This gives us the following result.

Result 3 The steady-state cartel price is decreasing in the but-for price, $\partial P^*/\partial \widehat{P} < 0$.

To understand this result, first note that lowering $\widehat{P}$ raises the total amount of damages by increasing the overcharge, which is the amount of damages assigned per unit of damage demand. One response to a lower but-for price is to lower the cartel price so as to bring back down the overcharge. Alternatively, firms could raise the cartel price so as to reduce the number of units upon which damages are assessed. Given α is not too low - so that the number of units used for the damage calculation is sufficiently sensitive to the collusive price - the latter effect dominates. Surprisingly, the steady-state cartel price is then decreasing in the but-for price. Thus, if the cartel anticipates that a more competitive standard will be applied in calculating damages, this will result in a *higher* cartel price in the long-run.

5 When Detection Depends on Both the Price Level and Price Changes

As argued at the start of Section 4, counterfactual results about the pattern of prices emerge when detection is assumed to depend only on the price level. More factual results follow when detection is driven by price changes. However, it is possible that detection is driven by both forces - being more likely when price changes are bigger and when price levels are higher. In this section, numerical analysis is used to explore such a possibility. As it turns out, allowing both a higher price level and bigger price changes to make detection more likely generates some possibly identifying pricing patterns to a price-fixing cartel.

Assume market demand is $1000 - P^t$ and each firm has constant marginal cost of zero. Parameter values are $n = 2, \delta = .75, \beta = .95, \gamma = 1$, and $F = 0$. The but-for price is assumed to be the Cournot price, which is 333, and the simple monopoly price is 500. The probability of detection is specified to be

$$\phi\left(P^t, P^{t-1}\right) = \min\left\{\phi_0 + \lambda\phi_1\left(P^t - \widehat{P}\right)^2 + (1-\lambda)\,\phi_2\left(P^t - P^{t-1}\right)^2, 1\right\}.$$

When $\lambda = 0$ then detection is only sensitive to price movements and $\lambda = 1$ results in detection depending only on the price level. ϕ_0 is set equal to .01 so that the baseline probability of detection is 1%. $\phi_1 = .00000324$ which implies that when $\lambda = 1$ then setting the monopoly price results in a 10% chance of detection and $\phi_2 = .00003204$ so that, when $\lambda = 0$, raising price from the non-collusive to the monopoly price in a single period results in a 90% chance of detection. The optimal price path was calculated for $\lambda \in \{0, .01, \ldots, .99, 1\}$. All of the resulting price paths can be found at

www.econ.jhu.edu/People/Harrington/cartelpricing.avi

where, by clicking the image, an animated movie shows how the price path changes when λ is raised from 0 to 1 so that the importance of the price level with regards to detection is increased relative to that of price changes.

Typical of our findings is Figure 2 which depicts the optimal cartel price path when $\lambda = .15$. The cartel begins by gradually raising price from 333 and thereby avoiding those big price changes that are likely to induce suspicions about collusion. Price peaks at 462 around period 10 after which it gradually declines and converges to its steady-state value of 448. I believe this pattern is quite general for the following reason. At the time of cartel formation, price is at its non-collusive level so that the task before the cartel is to raise price but to do so without triggering detection. This results in a gradual increase in price. As price tends towards its steady-state value, price changes are going to zero which means there is no first-order effect of price changes on detection. While, with price bounded above the non-collusive level, there *is* a first-order effect on detection from changing the price level. Hence, as price converges, the marginal impact of the price level on detection is becoming large relative to the marginal impact of price changes. The price path is then declining as the cartel seeks to lower the probability of detection with a lower price level and thus offset the fact that the penalty is increasing over time. By this logic, I then conjecture that a general prediction of allowing for detection to largely depend on price changes but also to be sensitive to the price level is that the price path will initially rise and then moderately decline to its steady-state value.

6 Concluding Remarks

In choosing a price path, it is natural to expect a price-fixing cartel to try to avoid creating suspicions that collusion is afoot. This paper is the first to explore how detection

impacts cartel pricing in the context of a dynamic model when detection and penalties are endogenous. Its contribution is two-fold. First, the intertemporal structure of the price path is characterized. When detection depends on price changes, the cartel gradually raises price. A more subtle property emerges when detection is driven by both price changes and price levels. In that case, the initial phase in which price is gradually increased is followed by having price moderately decline as it converges to its steady-state value. The second contribution is exploring the impact of antitrust policy on the steady-state price. Some of the ensuring results serve to alter our basic intuition. Based on static models, there is generally thought to be an equivalence between damages and (fixed) fines in the sense that if damages constrains the cartel to price at some level then there is a fixed fine that will do so as well. That equivalence breaks down in a dynamic model. When the only penalties are fines, the cartel's steady-state price is exactly the simple monopoly price. Antitrust policy fails to constrain the cartel. However, when damages are used, the steady-state price is below the simple monopoly price and, furthermore, is decreasing the damage multiple. A second accepted piece of intuition is that deployment of a more competitive standard for calculating damages will induce the cartel to price lower because now any given price has assigned to it a higher overcharge. I find just the contrary is true. A lower but-for price induces the cartel to price higher.

The model and analysis of this paper is an initial attempt to develop a richer dynamic theory of price-fixing cartels by taking account of their illegality and the desire of firms to avoid detection. There are many directions that one can go from here. With this particular model, there is a need to take account of (binding) equilibrium conditions so as to ensure that, more generally, firms do not want to deviate from the cartel price path. Of particular interest is to explore how antitrust policy interacts with these conditions. To what extent do concerns about detection make cheating more or less desirable and what is the role of antitrust policy in destabilizing the internal stability of cartels?

A second set of extensions is to encompass leniency programs. There have been a number of interesting papers exploring how leniency - in the form of allowing cartel members who provide evidence to receive reduced penalties - affects the degree of collusion and welfare. That work, however, does not take into account the endogeneity of detection and antitrust penalties. The central set of questions in that literature revolve around the optimal form of leniency. Should all firms be able to apply for leniency or just the first to come forward (as with the U.S. program)? While in both Europe and the U.S. leniency means avoidance of fines (also prison sentences in the U.S.), this leaves a firm still liable

for damages in the U.S. (and there are no damages in Europe). What difference does it make that some but not all penalties are avoided?

A third extension is related to the fact that detection has been assumed to depend only on movements in a common firm price. However, suspicions about collusion are also generated by firms' prices moving in tandem. If buyers may infer, rightly or wrongly, from parallel price movements that a cartel is present, this will also have implications for pricing behavior.

In conclusion, by taking into account the issue of detection, theory may eventually be able to empirically distinguish between explicit and tacit collusion. Tacit collusion I define as when firms engage in a pricing arrangement that serves to raise price and is achieved without explicit communication. While it is possible to prosecute tacitly colluding firms, it is very difficult. Explicit collusion is when firms engage in direct communication regarding the setting of prices (or some other form of collusion such as market allocation). Explicit collusion is clearly an antitrust violation. While antitrust case law makes a critical distinction between explicit and tacit collusion, existing collusive pricing theories do not.[18] However, if explicit collusion is illegal and tacit collusion is not (or at least it is considerably more difficult to prove illegality) then concerns about detection are much more important when firms have formed a price-fixing cartel (or what is called a "hard-core cartel" in policy circles). In the model of this paper, all pricing dynamics are driven by concerns about detection and penalties. Indeed, if tacit collusion is legal then the joint profit-maximizing price path under successful tacit collusion is to price at the simple monopoly price in all periods. This is strikingly different from the hard-core cartel price path in which price gradually rises and is bounded below the simple monopoly price. The qualitatively different pricing dynamics between explicit and tacit collusion offers some hope to distinguish between the two forms of collusion. This is quite important for policy purposes as it is best if the antitrust authority allocates its resources to prosecuting explicit collusion for there is both more hope of achieving a conviction and in deterring the formation of hard-core cartels.

[18] There are a few exceptions. McCutcheon (1997) models meetings between firms. Athey, Bagwell, and Sanchirico (1998) and Athey and Bagwell (2001) model the exchange of cost information by firms which would seem more appropriate for explicit than tacit collusion (though such exchange could still occur through a trade association).

Appendix A

Proof of Theorem 1 The proof is an adaptation of arguments in Stokey and Lucas (1989). Begin by supposing that the cartel has been formed and let $v : \Omega \times \left[0, \overline{X}\right] \to \Re$ be a continuous (and necessarily bounded) function. Note that the boundedness of x and $\beta < 1$ imply X^t is bounded and we let $\overline{X}$ denote such a bound. Let T be a function with domain B which is the space of continuous functions that map $\Omega \times \left[0, \overline{X}\right]$ into $\Re$. T is defined as follows:

$$\begin{aligned} T\left(v\left(\cdot\right)\right) &= \max_{P \in \Omega} \pi\left(P\right) + \delta\phi\left(P, P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta X^{t-1} - \gamma x\left(P\right) - F\right] \\ &\quad + \delta\left[1 - \phi\left(P, P^{t-1}\right)\right] \max\{v\left(P, \beta X^{t-1} + \gamma x\left(P\right)\right), \\ &\quad \left(\widehat{\pi}/\left(1-\delta\right)\right) - \sigma\left(P, \beta X^{t-1} + \gamma x\left(P\right)\right)\}. \end{aligned} \tag{11}$$

By A1-A5 and the presumption that v is a continuous function, the above problem involves maximizing a continuous function over a compact set. Hence, $T\left(v\left(\cdot\right)\right)$ exists by the Theorem of the Maximum (Theorem 3.6, Stokey and Lucas, 1989). As π, ϕ, x, σ, and v are continuous functions and Ω is compact, T is a continuous function (Theorem 3.6, Stokey and Lucas, 1989). Hence, the range of T is B so that $T : B \to B$.

To show that T is a contraction, Blackwell's theorem is used (Theorem 3.3, Stokey and Lucas, 1989). This requires showing that T satisfies monotonicity and discounting. Monotonicity is satisfied when: if $v^o, v^{oo} \in B$ and

$$v^o\left(P^{t-1}, X^{t-1}\right) \leq v^{oo}\left(P^{t-1}, X^{t-1}\right) \forall \left(P^{t-1}, X^{t-1}\right) \in \Omega \times \left[0, \overline{X}\right]$$

then

$$T\left(v^o\left(P^{t-1}, X^{t-1}\right)\right) \leq T\left(v^{oo}\left(P^{t-1}, X^{t-1}\right)\right) \forall \left(P^{t-1}, X^{t-1}\right) \in \Omega \times \left[0, \overline{X}\right].$$

This is trivially true. Discounting is satisfied when $\exists \theta \in (0,1)$ such that

$$\begin{aligned} T\left(v\left(P^{t-1}, X^{t-1}\right) + a\right) &\leq T\left(v\left(P^{t-1}, X^{t-1}\right)\right) + \theta a \\ \forall v &\in B, a \geq 0, \left(P^{t-1}, X^{t-1}\right) \in \Omega \times \left[0, \overline{X}\right]. \end{aligned}$$

First note that

$$\begin{aligned}
& T\left(v\left(P^{t-1},X^{t-1}\right)+a\right) \\
= & \max_{P}\pi\left(P\right)+\delta\phi\left(P,P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta X^{t-1}-\gamma x\left(P\right)-F\right] \\
& +\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{v^{o}\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right)+a, \\
& \left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right)\} \\
\leq & \max_{P}\pi\left(P\right)+\delta\phi\left(P,P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta X^{t-1}-\gamma x\left(P\right)-F\right] \\
& +\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{v^{o}\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right), \\
& \left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right)\}+\delta\left[1-\phi\left(P,P^{t-1}\right)\right]a \\
\leq & \max_{P}\pi\left(P\right)+\delta\phi\left(P,P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta X^{t-1}-\gamma x\left(P\right)-F\right] \\
& +\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{v^{o}\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right), \\
& \left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right)\}+\delta a \\
= & T\left(v\left(P^{t-1},X^{t-1}\right)\right)+\delta a.
\end{aligned}$$

As $\delta\in(0,1)$, T is a contraction. Since the space of continuous functions over a compact subset of Euclidean space is a complete metric space (in the sup metric) then, by the Contraction Mapping Theorem (Theorem 3.2, Stokey and Lucas, 1989), T has a unique fixed point which is a continuous function. This fixed point is the value function, V. Since then

$$\begin{aligned}
& \pi\left(P\right)+\delta\phi\left(P,P^{t-1}\right)\left[\left(\widehat{\pi}/\left(1-\delta\right)\right)-\beta X^{t-1}-\gamma x\left(P\right)-F\right] \\
& +\delta\left[1-\phi\left(P,P^{t-1}\right)\right]\max\{V\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right), \\
& \left(\widehat{\pi}/\left(1-\delta\right)\right)-\sigma\left(P,\beta X^{t-1}+\gamma x\left(P\right)\right)\}
\end{aligned} \tag{12}$$

is a continuous function and Ω is compact, an optimal price path exists.

All of this analysis is for when the cartel has been formed. If $V\left(P^{0},X^{0}\right)\geq\widehat{\pi}/\left(1-\delta\right)$ then it is indeed optimal to form the cartel and the price path is that which maximizes (12). If $V\left(P^{0},X^{0}\right)<\widehat{\pi}/\left(1-\delta\right)$ then it is not optimal to form the cartel and the optimal price path is $\widehat{P}$ in all periods. ■

Proof of Theorem 2 Define the sequence of value functions $\{v_h(\cdot)\}_{h=1}^{\infty}$:

$$\begin{aligned} v_{h+1}\left(P^{t-1}, X^{t-1}\right) &= \max_{P\in\Omega} \pi(P) \\ &\quad +\delta\phi\left(P, P^{t-1}\right)\left[(\widehat{\pi}/(1-\delta)) - \beta X^{t-1} - \gamma x(P) - F\right] \\ &\quad +\delta\left[1-\phi\left(P, P^{t-1}\right)\right]\max\{v_h\left(P, \beta X^{t-1} + \gamma x(P)\right), \\ &\quad (\widehat{\pi}/(1-\delta)) - \sigma\left(P, \beta X^{t-1} + \gamma x(P)\right)\}. \end{aligned} \tag{13}$$

Given A1-A5, it follows from the Contraction Mapping Theorem that $V(\cdot) = \lim_{h\to\infty} v_h(\cdot)$. Any property that holds for the sequence of functions $\{v_h(\cdot)\}_{h=1}^{\infty}$ then holds for $V(\cdot)$.

Suppose $v_h(\cdot)$ is decreasing in X^{t-1}. Since $\sigma(\cdot)$ is nondecreasing in X^{t-1} then $\max\{v_h(\cdot), (\widehat{\pi}/(1-\delta)) - \sigma(\cdot)\}$ is nondecreasing in X^{t-1}. Using this fact along with $\left[(\widehat{\pi}/(1-\delta)) - \beta X^{t-1} - \gamma x(P) - F\right]$ being decreasing in X^{t-1}, it follows that $v_{h+1}(\cdot)$ is decreasing in X^{t-1}. Therefore, $V(\cdot)$ is decreasing in X^{t-1}.

Suppose $v_h(\cdot)$ is convex in X^{t-1}. As $\sigma(\cdot)$ is concave in X^{t-1} and, since the maximum of convex functions is convex, then $\max\{v_h(\cdot), (\widehat{\pi}/(1-\delta)) - \sigma(\cdot)\}$ is convex in X^{t-1}.[19] Next note that $\left[(\widehat{\pi}/(1-\delta)) - \beta X^{t-1} - \gamma x(P) - F\right]$ is convex in X^{t-1}. Thus, the function being maximized on the rhs of (13) is convex in X^{t-1}. As $v_{h+1}(\cdot)$ is simply the maximum of a collection of convex functions - each one parameterized by a different element of Ω - then $v_{h+1}(\cdot)$ is convex in X^{t-1}. ■

Proof of Theorem 3 There are several steps in the proof. First, it is shown that if it is optimal to form a cartel then it is optimal to collude forever. Second, the optimal price path is bounded above by P^*. Third, the optimal price path is non-decreasing over time. Fourth, the optimal price path converges to P^*.

- It is optimal to collude forever.

The strategy is to show that if it is optimal to collude in, say, period T then it must be optimal to collude in period $T+1$. Assume it is optimal to form a cartel. It is sufficient to show that it is optimal to collude forever when $\sigma\left(P^{t-1}, X^{t-1}\right) = 0 \forall \left(P^{t-1}, X^{t-1}\right)$ so that the terminal payoff from stopping collusion is $\widehat{\pi}/(1-\delta)$. Suppose it is optimal to

[19] Suppose $u_1(\cdot), u_2(\cdot), \ldots, u_k(\cdot)$ are convex in z. To show that $U(\cdot) \equiv \max\{u_1(\cdot), u_2(\cdot), \ldots, u_k(\cdot)\}$ is convex, suppose to the contrary. This means that $\exists z', z''$ and $\lambda \in (0,1)$ such that $\lambda U(z') + (1-\lambda)U(z'') < U(\lambda z' + (1-\lambda)z'')$. Suppose $U(z') = u_i(z')$, $U(z'') = u_j(z'')$, and $U(\lambda z' + (1-\lambda)z'') = u_k(\lambda z' + (1-\lambda)z'')$. The condition is then: $\lambda u_i(z') + (1-\lambda)u_j(z'') < u_k(\lambda z' + (1-\lambda)z'')$. Since $u_i(z') \geq u_k(z')$ and $u_j(z'') \geq u_k(z'')$, it follows that: $\lambda u_k(z') + (1-\lambda)u_k(z'') < u_k(\lambda z' + (1-\lambda)z'')$, but this contradicts the assumption that $u_k(\cdot)$ is convex.

collude until period T where T is finite. For it to be optimal to collude in T, it must be true that:

$$\pi\left(P^T\right) - \delta\phi\left(P^T, P^{T-1}\right)\left[\beta X^{T-1} + \gamma x\left(P^T\right) + F\right] + \frac{\delta\widehat{\pi}}{1-\delta} \geq \frac{\widehat{\pi}}{1-\delta}.$$

The lhs is the payoff from colluding in T and stopping collusion as of $T+1$ and the rhs is the payoff from stopping collusion in T. This expression is equivalent to:

$$\pi\left(P^T\right) - \delta\phi\left(P^T, P^{T-1}\right)\left[\beta X^{T-1} + \gamma x\left(P^T\right) + F\right] \geq \widehat{\pi}. \tag{14}$$

For it to be optimal to dismantle the cartel in $T+1$, it is necessary that:

$$\begin{aligned}\frac{\widehat{\pi}}{1-\delta} &> \pi\left(P^T\right) - \delta\widehat{\phi}\left(0\right)\left[\beta\left(\beta X^{T-1} + \gamma x\left(P^T\right)\right) + \gamma x\left(P^T\right) + F\right] + \frac{\delta\widehat{\pi}}{1-\delta} \Leftrightarrow \\ \widehat{\pi} &> \pi\left(P^T\right) - \delta\widehat{\phi}\left(0\right)\left[\beta\left(\beta X^{T-1} + \gamma x\left(P^T\right)\right) + \gamma x\left(P^T\right) + F\right]. \end{aligned} \tag{15}$$

The rhs of the first line in (15) is the payoff from maintaining a price of P^T in $T+1$ and then stopping collusion as of $T+2$.[20] Note that $\phi\left(P^T, P^T\right) = \widehat{\phi}\left(0\right)$. Combining (14)-(15):

$$\begin{aligned} & \pi\left(P^T\right) - \delta\phi\left(P^T, P^{T-1}\right)\left[\beta X^{T-1} + \gamma x\left(P^T\right) + F\right] \\ \geq\ & \widehat{\pi} > \pi\left(P^T\right) - \delta\widehat{\phi}\left(0\right)\left[\beta\left(\beta X^{T-1} + \gamma x\left(P^T\right)\right) + \gamma x\left(P^T\right) + F\right]. \end{aligned}$$

A necessary condition for this to hold is:

$$\begin{aligned} & \pi\left(P^T\right) - \delta\phi\left(P^T, P^{T-1}\right)\left[\beta X^{T-1} + \gamma x\left(P^T\right) + F\right] \\ >\ & \pi\left(P^T\right) - \delta\widehat{\phi}\left(0\right)\left[\beta\left(\beta X^{T-1} + \gamma x\left(P^T\right)\right) + \gamma x\left(P^T\right) + F\right] \end{aligned}$$

or

$$\widehat{\phi}\left(0\right)\left[\beta\left(\beta X^{T-1} + \gamma x\left(P^T\right)\right) + \gamma x\left(P^T\right) + F\right] > \phi\left(P^T, P^{T-1}\right)\left[\beta X^{T-1} + \gamma x\left(P^T\right) + F\right].$$

Since, by A8, $\phi\left(P^T, P^{T-1}\right) \geq \widehat{\phi}\left(0\right)$, a necessary condition is:

$$\beta\left(\beta X^{T-1} + \gamma x\left(P^T\right)\right) + \gamma x\left(P^T\right) > \beta X^{T-1} + \gamma x\left(P^T\right) \Leftrightarrow \frac{\gamma x\left(P^T\right)}{(1-\beta)} > X^{T-1}.$$

Intuitively, if it is optimal to collude at a price of P^T in period T but it is not optimal to do so in $T+1$ then damages must be higher in $T+1$. For that to be the case, what is added to damages in T, $\gamma x\left(P^T\right)$, must exceed the amount of damages lost through depreciation, $(1-\beta)X^{T-1}$. This produces the above condition.

[20] The assumption is used that a firm must strictly prefer not to collude for it to dissolve the cartel.

Next note that it is never optimal for the cartel price to exceed the simple monopoly price of P^m. Relative to a price of P^m, a higher price yields strictly lower current profit, weakly higher damages, and, as price initially starts below P^m, a weakly higher probability of detection. It is straightforward to show that a price path with prices above P^m yields a lower payoff to one in which all those prices exceeding P^m are replaced with P^m. Since then $P^T \leq P^m$, it follows from A10 that:

$$\pi\left(P^T\right) - \delta\widehat{\phi}\left(0\right)\left[\left(\frac{\gamma x\left(P^T\right)}{1-\beta}\right) + F\right] > \widehat{\pi}. \tag{16}$$

Given it has been shown that X^{T-1} is bounded above by $\gamma x\left(P^T\right)/\left(1-\beta\right)$, (16) contradicts (15). This contradiction establishes that the claim that collusion stops in finite time is false.

- The optimal price path is bounded above by P^*.

The proof strategy is to show that if the price path ever exceeds P^* that a higher payoff is realized by pricing at P^* forever, starting in the period with which price first exceeds P^*.

Assuming firms collude forever and using the representation of the payoff in (25), the payoff starting from period t' for the collusive price path $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right) - \overline{\Delta}^{t'}\gamma x\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \overline{\Delta}^{t'}\beta X^{t'-1} \\
&+ \sum_{t=t'+1}^{\infty}\delta^{t-t'}\left\{\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^t\right) - \overline{\Delta}^t\gamma x\left(\overline{P}^t\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\right\} \\
&+ \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]
\end{aligned} \tag{17}$$

where

$$\overline{\Delta}^t \equiv \delta\sum_{\tau=t}^{\infty}\left(\delta\beta\right)^{\tau-t}\phi\left(\overline{P}^{\tau},\overline{P}^{\tau-1}\right)\prod_{j=t}^{\tau-1}\left[1-\phi\left(\overline{P}^j,\overline{P}^{j-1}\right)\right].$$

In considering (17), it is as if a colluding firm receives net income in each period equal to $\pi\left(\overline{P}^t\right) - \overline{\Delta}^t\gamma x\left(\overline{P}^t\right)$ where $\pi\left(\overline{P}^t\right)$ is gross profit and $\overline{\Delta}^t\gamma x\left(\overline{P}^t\right)$ is the expected present value of damages associated with colluding in that period.

Suppose it is not true that price is bounded above by P^* so $\exists t'$ such that $\overline{P}^{t'} > P^* \geq \overline{P}^{t'-1}$. If this price path is optimal then, starting from period t', it must yield at least as high a payoff as a price path in which firms collude and price at P^* forever. This is true

iff:

$$\begin{aligned}
&\left[\pi\left(\overline{P}^{t'}\right)-\overline{\Delta}^{t'}\gamma x\left(\overline{P}^{t'}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]-\overline{\Delta}^{t'}\beta X^{t'-1} \qquad (18)\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\left\{\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\overline{\Delta}^{t}\gamma x\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]\right\}\\
\geq\ &\left[\pi\left(P^{*}\right)-\widetilde{\Delta}^{t'}\gamma x\left(P^{*}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]-\widetilde{\Delta}^{t'}\beta X^{t'-1}\\
&+\sum_{t=t'+1}^{\infty}\delta^{t-t'}\left\{\left[1-\phi\left(P^{*},\overline{P}^{t'-1}\right)\right]\left[1-\widehat{\phi}\left(0\right)\right]^{t-t'-1}\left[\pi\left(P^{*}\right)-\widetilde{\Delta}^{t}\gamma x\left(P^{*}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]\right\}
\end{aligned}$$

where

$$\widetilde{\Delta}^{t'}\equiv\delta\left\{\phi\left(P^{*},\overline{P}^{t'-1}\right)+\sum_{\tau=t'+1}^{\infty}(\delta\beta)^{\tau-t'}\left[1-\phi\left(P^{*},\overline{P}^{t'-1}\right)\right]\left[1-\widehat{\phi}\left(0\right)\right]^{\tau-t'-1}\widehat{\phi}\left(0\right)\right\},$$

$$\widetilde{\Delta}^{t}\equiv\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\left[1-\widehat{\phi}\left(0\right)\right]^{\tau-t}\widehat{\phi}\left(0\right),\ t\geq t'+1$$

and recall that $\phi\left(P^{*},P^{*}\right)=\widehat{\phi}\left(0\right)$. To show that $\overline{\Delta}^{t}\geq\widetilde{\Delta}^{t}\forall t\geq t'$, first note that these expressions can be represented as:

$$\Delta^{t}=\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\omega^{\tau}\prod_{j=t}^{\tau-1}\left(1-\omega^{j}\right)$$

where ω^{τ} is the probability of detection in period τ condition on no detection as of $\tau-1$. Note that:

$$\begin{aligned}
\frac{\partial\Delta^{t}}{\partial\omega^{t^{o}}} &= \delta\left\{(\delta\beta)^{t^{o}-t}\prod_{j=t}^{t^{o}-1}\left(1-\omega^{j}\right)-\sum_{\tau=t^{o}+1}^{\infty}(\delta\beta)^{\tau-t}\omega^{\tau}\prod_{j=t,j\neq t^{o}}^{\tau-1}\left(1-\omega^{j}\right)\right\}\\
&= \delta\left(\delta\beta\right)^{t^{o}-t}\prod_{j=t}^{t^{o}-1}\left(1-\omega^{j}\right)\left\{1-\sum_{\tau=t^{o}+1}^{\infty}(\delta\beta)^{\tau-t^{o}}\omega^{\tau}\prod_{j=t^{o}+1}^{\tau-1}\left(1-\omega^{j}\right)\right\}.
\end{aligned}$$

$\sum_{\tau=t^{o}+1}^{\infty}\omega^{\tau}\prod_{j=t^{o}+1}^{\tau-1}\left(1-\omega^{j}\right)$ is the probability of detection over periods $t^{o}+1,\ldots,\infty$. Since it is less than or equal to one, it follows that:

$$1-\sum_{\tau=t^{o}+1}^{\infty}(\delta\beta)^{\tau-t^{o}}\omega^{\tau}\prod_{j=t^{o}+1}^{\tau-1}\left(1-\omega^{j}\right)>0.$$

Thus, Δ^{t} is increasing in $\omega^{t^{o}}$. Since $\overline{P}^{t'}>P^{*}\geq\overline{P}^{t'-1}$ then, by A7, $\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(P^{*},\overline{P}^{t'-1}\right)$. By A8, $\phi\left(\overline{P}^{t},\overline{P}^{t-1}\right)\geq\widehat{\phi}\left(0\right),\ t\geq t'+1$. The probability of detection in period τ (condition on no detection as of $\tau-1$) is then weakly higher for $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$ than for the alternative price path $\forall t\geq t'$. It is concluded that $\overline{\Delta}^{t}\geq\widetilde{\Delta}^{t}\forall t\geq t'$.

Consider the lhs expression in (18). Since it is non-increasing in $\overline{\Delta}^t$ and $\overline{\Delta}^t \geq \widetilde{\Delta}^t \forall t \geq t'$, the expression is weakly increased if $\widetilde{\Delta}^t$ replaces $\overline{\Delta}^t$ $\forall t \geq t'$. It follows that if (18) holds then it must be true that:

$$\begin{aligned} & \left[\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}^{t'}\gamma x\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \widetilde{\Delta}^{t'}\beta X^{t'-1} \\ & + \sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \widetilde{\Delta}^{t}\gamma x\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] \right\} \\ \geq & \left[\pi\left(P^*\right) - \widetilde{\Delta}^{t'}\gamma x\left(P^*\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \widetilde{\Delta}^{t'}\beta X^{t'-1} \\ & + \sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right] \left[1 - \widehat{\phi}\left(0\right)\right]^{t-t'-1} \left[\pi\left(P^*\right) - \widetilde{\Delta}^{t}\gamma x\left(P^*\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] \right\}. \end{aligned} \tag{19}$$

The objective is to establish that a contradiction follows from (19). The first step is to show that the summation term on the rhs is at least as great as the summation term on the lhs. As $\widetilde{\Delta}^t = \delta\widehat{\phi}\left(0\right) / \left[1 - \delta\beta\left(1 - \widehat{\phi}\left(0\right)\right)\right]$, it follows from A11 that

$$\pi\left(P^*\right) - \widetilde{\Delta}^{t}\gamma x\left(P^*\right) \geq \pi\left(\overline{P}^{t}\right) - \widetilde{\Delta}^{t}\gamma x\left(\overline{P}^{t}\right),\ t \geq t'+1.$$

Given $\delta\widehat{\phi}\left(0\right)/\left(1-\beta\right) \geq \widetilde{\Delta}^t$, A10 implies $\pi\left(P^*\right) - \widetilde{\Delta}^{t}\gamma x\left(P^*\right) > \widehat{\pi} + \delta\widehat{\phi}\left(0\right)F$ and thus $\pi\left(P^*\right) - \widetilde{\Delta}^{t}\gamma x\left(P^*\right) > \widehat{\pi} - (1-\delta)F$. Finally, note that

$$\left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right] \left[1 - \widehat{\phi}\left(0\right)\right]^{t-t'-1} \geq \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right],\ t \geq t'+1.$$

It is concluded that

$$\begin{aligned} & \sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \widetilde{\Delta}^{t}\gamma x\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] \right\} \\ \leq & \sum_{t=t'+1}^{\infty} \delta^{t-t'} \left\{ \left[1 - \phi\left(P^*, \overline{P}^{t'-1}\right)\right] \left[1 - \widehat{\phi}\left(0\right)\right]^{t-t'-1} \left[\pi\left(P^*\right) - \widetilde{\Delta}^{t}\gamma x\left(P^*\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] \right\}. \end{aligned}$$

Thus, (19) implies:

$$\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}^{t'}\gamma x\left(\overline{P}^{t'}\right) \geq \pi\left(P^*\right) - \widetilde{\Delta}^{t'}\gamma x\left(P^*\right). \tag{20}$$

Since $\gamma x\left(\overline{P}^{t'}\right) \geq \gamma x\left(P^*\right)$ (so that the lhs is decreasing in $\widetilde{\Delta}^{t'}$ at a faster rate than the rhs), it follows from $\widetilde{\Delta}^{t'} \geq \widetilde{\Delta}^{t}$ that (20) implies:

$$\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}^{t}\gamma x\left(\overline{P}^{t'}\right) \geq \pi\left(P^*\right) - \widetilde{\Delta}^{t}\gamma x\left(P^*\right).$$

Since $\widetilde{\Delta}^t = \delta\widehat{\phi}\left(0\right) / \left[1 - \delta\beta\left(1 - \widehat{\phi}\left(0\right)\right)\right]$ and $\overline{P}^{t'} > P^*$, this cannot be true by A11. This proves that the price path is bounded above by P^*.

- The optimal price path is non-decreasing over time.

The proof strategy involves two parts. First, suppose that price falls from $t'-1$ to t' and furthermore that price never exceeds its level prior to the decline, that is, $P^{t'-1} \geq P^t \forall t \geq t'$. It is shown that a higher payoff is realized when price is kept constant at $P^{t'-1} \forall t \geq t'$. Second, suppose that price falls from $t'-1$ to t' and remains at or below $P^{t'-1}$ over periods $t'+1, ..., t''$. It is then shown that a higher payoff is realized by skipping the price path over periods $t'+1, ..., t''$ and jumping to a price of $P^{t''+1}$ in period t', $P^{t''+2}$ in period $t'+1$, and so forth.

Suppose $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an optimal price path and it is not non-decreasing over time. Hence, $\exists t' > 1$ such that $P^0 < \overline{P}^1 \leq \cdots \leq \overline{P}^{t'-1} > \overline{P}^{t'}$. A necessary condition for optimality is that the payoff, starting in t', from $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is at least as great as maintaining price at $\overline{P}^{t'-1}$ forever:

$$\begin{aligned}
& \left[\pi\left(\overline{P}^{t'}\right) - \overline{\Delta}^{t'}\gamma x\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \overline{\Delta}^{t'}\beta X^{t'-1} \qquad (21)\\
& + \sum_{t=t'+1}^{\infty}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^t\right) - \overline{\Delta}^t\gamma x\left(\overline{P}^t\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\\
& + \left[\widehat{\pi}/(1-\delta) - F\right]\\
\geq\ & \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \widetilde{\Delta}\beta X^{t'-1}\\
& + \sum_{t=t'+1}^{\infty}\delta^{t-t'}\left[1-\widehat{\phi}(0)\right]^{t-t'}\left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\\
& + \left[\widehat{\pi}/(1-\delta) - F\right]
\end{aligned}$$

$$\text{where } \widetilde{\Delta} \equiv \delta\sum_{\tau=t}^{T}(\delta\beta)^{\tau-t}\left[1-\widehat{\phi}(0)\right]^{\tau-t}\widehat{\phi}(0).$$

The first step is to show that if $\overline{P}^{t'-1} > \overline{P}^{t'}$ and $\overline{P}^{t'-1} \geq \overline{P}^t \forall t \geq t'+1$ then (21) cannot be true; maintaining price at $\overline{P}^{t'-1}$ forever is superior. Recall that price is bounded above by P^* so that $\overline{P}^{t'-1} \leq P^*$. Since $\widetilde{\Delta} \leq \overline{\Delta}^t \forall t$ then the lhs of (21) is less than:

$$\begin{aligned}
& \left[\pi\left(\overline{P}^{t'}\right) - \widetilde{\Delta}\gamma x\left(\overline{P}^{t'}\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] - \widetilde{\Delta}\beta X^{t'-1}\\
& + \sum_{t=t'+1}^{\infty}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^t\right) - \widetilde{\Delta}\gamma x\left(\overline{P}^t\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right]\\
& + \left[\widehat{\pi}/(1-\delta) - F\right].
\end{aligned}$$

Hence, a necessary condition for (21) to be true is:

$$\begin{aligned} & \left[\pi\left(\overline{P}^{t'}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right] \\ & +\sum_{t=t'+1}^{\infty}\delta^{t-t'}\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right]\left[\pi\left(\overline{P}^{t}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right] \\ \geq & \left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right] \\ & +\sum_{t=t'+1}^{\infty}\delta^{t-t'}\left[1-\widehat{\phi}(0)\right]^{t-t'}\left[\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right] \end{aligned} \tag{22}$$

To show that the summation term on the rhs is at least as great as that on the lhs, first note that A11 implies

$$\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-(1-\delta)F\right)\geq\pi\left(\overline{P}^{t}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-(1-\delta)F\right),\ t\geq t'+1.$$

as, by supposition, $\overline{P}^{t'-1}\geq\overline{P}^{t}\forall t\geq t'+1$ and it has already been proven that $P^{*}\geq\overline{P}^{t'-1}$. Next note that A10 implies

$$\begin{aligned} \pi\left(\overline{P}^{t}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t}\right)-\left(\widehat{\pi}-(1-\delta)F\right) &> 0 \text{ and} \\ \pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right)-\left(\widehat{\pi}-(1-\delta)F\right) &> 0, \end{aligned}$$

$\forall t\geq t'+1$ because $\overline{P}^{t'-1},\overline{P}^{t}\leq P^{m}$ and $\widetilde{\Delta}\leq\delta\widehat{\phi}(0)/(1-\beta)$. Finally,

$$\left[1-\widehat{\phi}(0)\right]^{t-t'}\geq\prod_{j=t'}^{t-1}\left[1-\phi\left(\overline{P}^{j},\overline{P}^{j-1}\right)\right],\ t\geq t'+1.$$

It is concluded that the summation term on the rhs of (22) is at least as great as the summation term on the lhs of (22). Therefore, for (22) (and hence, (21)) to be true, it is necessary that:

$$\pi\left(\overline{P}^{t'}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'}\right)\geq\pi\left(\overline{P}^{t'-1}\right)-\widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right).$$

However, by $\overline{P}^{t'}<\overline{P}^{t'-1}\leq P^{*}$, this contradicts A11. It is concluded that the price path cannot be bounded above by $\overline{P}^{t'-1}$ for $t\geq t'$.

Therefore, if $\overline{P}^{t'-1}>\overline{P}^{t'}$ then $\exists t''\geq t'$ such that $\overline{P}^{t'-1}\geq\overline{P}^{t'+1},\ldots,\overline{P}^{t''}$ and $\overline{P}^{t'-1}<\overline{P}^{t''+1}$. Once again compare this price path with one in which price is kept constant at $\overline{P}^{t'-1}$. By the arguments just given, one can show that the income from $\left\{\overline{P}^{t}\right\}_{t=1}^{\infty}$ is strictly lower at t' and is weakly lower at periods $t'+1,\ldots,t''$. Hence, a necessary condition for

optimality is that the sum of the discounted terms for periods $t \geq t''+1$ is strictly higher:

$$\begin{aligned} & \sum_{t=t''+1}^{\infty} \delta^{t-t'} \prod_{j=t'}^{t-1} \left[1-\phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \times \\ & \left[\pi\left(\overline{P}^{t}\right) - \overline{\Delta}^{t} \gamma x\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \\ > & \sum_{t=t''+1}^{\infty} \delta^{t-t'} \left[1-\widehat{\phi}(0)\right]^{t-t'} \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta} \gamma x\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \end{aligned} \tag{23}$$

or

$$\begin{aligned} & \delta^{t''-t'+1} \prod_{j=t'}^{t''} \left[1-\phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \times \\ & \sum_{t=t''+1}^{\infty} \delta^{t-t''-1} \prod_{j=t''+1}^{t-1} \left[1-\phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \overline{\Delta}^{t} \gamma x\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \\ > & \delta^{t''-t'+1} \left[1-\widehat{\phi}(0)\right]^{t''-t'+1} \times \\ & \sum_{t=t''+1}^{\infty} \delta^{t-t''-1} \left[1-\widehat{\phi}(0)\right]^{t-t''-1} \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta} \gamma x\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right]. \end{aligned}$$

Since

$$\theta \equiv \delta^{t''-t'+1} \left[1-\widehat{\phi}(0)\right]^{t''-t'+1} \geq \delta^{t''-t'+1} \prod_{j=t'}^{t''} \left[1-\phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \equiv \xi$$

then a necessary condition for (23) is:

$$\begin{aligned} Y \equiv & \sum_{t=t''+1}^{\infty} \delta^{t-t''-1} \prod_{j=t''+1}^{t-1} \left[1-\phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \overline{\Delta}^{t} \gamma x\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \\ > & \sum_{t=t''+1}^{\infty} \delta^{t-t''-1} \left[1-\widehat{\phi}(0)\right]^{t-t''-1} \times \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta} \gamma x\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right] \equiv X. \end{aligned}$$

From this condition it will be argued that a strictly superior price path to $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ is to set $P^{t} = \overline{P}^{t+t''-t'+1}$, $t \geq t'$. The reason is simple. It has been shown that $\left\{\overline{P}^{t}\right\}_{t=t'}^{t''}$ does worse than a constant price of $\overline{P}^{t'-1}$ over periods $t', \ldots, t''$. The optimality of $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ then requires that a strictly higher payoff be received after t''. Beginning from t', a higher payoff to $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ can then be earned by skipping the prices over $t', \ldots, t''$ and start pricing in t' according to the price path as of $t''+1$.

Define y and z as the payoff over $t', \ldots, t''$ from the price path $\left\{\overline{P}^{t}\right\}_{t=t'}^{\infty}$ and a constant price of $\overline{P}^{t'-1}$, respectively,

$$y \equiv \sum_{t=t'}^{t''} \delta^{t-t'} \prod_{j=t'}^{t-1} \left[1-\phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \left[\pi\left(\overline{P}^{t}\right) - \overline{\Delta}^{t} \gamma x\left(\overline{P}^{t}\right) - \left(\widehat{\pi} - (1-\delta) F\right)\right],$$

$$z \equiv \sum_{t=t'}^{t''} \delta^{t-t'} \left[1-\widehat{\phi}\left(0\right)\right]^{t-t'} \left[\pi\left(\overline{P}^{t'-1}\right) - \widetilde{\Delta}\gamma x\left(\overline{P}^{t'-1}\right) - \left(\widehat{\pi} - \left(1-\delta\right)F\right)\right].$$

Note that $Z = z/\left(1-\theta\right)$. In this notation, (21) takes the form:

$$y + \xi Y - \overline{\Delta}^{t'}\beta X^{t'-1} + \left[\widehat{\pi}/\left(1-\delta\right) - F\right] \geq z + \theta Z - \widetilde{\Delta}\beta X^{t'-1} + \left[\widehat{\pi}/\left(1-\delta\right) - F\right].$$

Consider:

$$Y - \left(y + \xi Y\right) = \left(1-\xi\right)Y - y > \left(1-\xi\right)Y - z = \left(1-\xi\right)Y - \left(1-\theta\right)Z > 0.$$

The last inequality follows from $\theta \geq \xi$ and that it has been shown that (21) implies $Y > Z$. It is then true that: $Y > y + \xi Y$. Now consider the payoff starting from t' in which $P^t = \overline{P}^{t+t''-t'+1}$, $t \geq t'$. It will be shown that it is bounded below by $Y - \overline{\Delta}^{t'}\beta X^{t'-1} + \left[\widehat{\pi}/\left(1-\delta\right) - F\right]$. As defined, Y is the payoff from $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$ starting in $t''+1$ and discounting back to $t''+1$ with an initial price of $\overline{P}^{t''}$. It is also the payoff from $P^t = \overline{P}^{t+t''-t'+1}$ for $t \geq t'$, starting in t' and discounting back to t' but with one caveat. The preceding price to $\overline{P}^{t''+1}$ is not $\overline{P}^{t''}$ but rather $\overline{P}^{t'-1}$. Since $\overline{P}^{t''+1} > \overline{P}^{t'-1} \geq \overline{P}^{t''}$ then

$$\left(\overline{P}^{t''+1} - \overline{P}^{t''}\right) g\left(\overline{P}^{t''}\right) \geq \left(\overline{P}^{t''+1} - \overline{P}^{t'-1}\right) g\left(\overline{P}^{t'-1}\right) > 0,$$

so that, by A7, the probability of detection at t' from the price path $P^t = \overline{P}^{t+t''-t'+1}$ is no greater than that at $t''+1$ from $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$.[21] Thus, the associated payoff is weakly higher than $Y - \overline{\Delta}^{t'}\beta X^{t'-1} + \left[\widehat{\pi}/\left(1-\delta\right) - F\right]$.

To summarize, it has been shown that a price path of $P^t = \overline{P}^{t+t''-t'+1}$ for $t \geq t'$ yields a payoff of at least $Y - \overline{\Delta}^{t'}\beta X^{t'-1} + \left[\widehat{\pi}/\left(1-\delta\right) - F\right]$ while $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$ yields a payoff of $y + \xi Y - \overline{\Delta}^{t'}\beta X^{t'-1} + \left[\widehat{\pi}/\left(1-\delta\right) - F\right]$. Since $Y > y + \xi Y$ then the former is larger which contradicts the optimality of $\left\{\overline{P}^t\right\}_{t=t'}^{\infty}$. This contradiction shows the falsity of the supposition that $\exists t' > 1$ such that $P^0 < P^1 \leq \cdots \leq \overline{P}^{t'-1} > \overline{P}^{t'}$. It is concluded that the price path is non-decreasing.

- The optimal price path converges to P^*.

A variational approach is used to characterize the limiting price. If $\left\{\overline{P}^t\right\}_{t=1}^{\infty}$ is an optimal price path then it is non-decreasing and is bounded above by P^*. Therefore, $\lim_{t\to\infty} P^t$ exists and is denoted $\overline{P}$. Consider a price path in which $P^t = \overline{P}^t$ for $t < T$

[21] This is the only step in the proof that requires g to be a non-increasing function.

and $P^t = \overline{P}^t + \varepsilon$ for $t \geq T$. Starting with period T, it yields a payoff of

$$\begin{aligned}
&\pi\left(\overline{P}^T + \varepsilon\right) \\
&- \left\{\delta\phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right) + \delta\beta\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right]\overline{\Delta}^{T+1}\right\} \times \\
&\left[\gamma x\left(\overline{P}^T + \varepsilon\right) + \beta X^{T-1}\right] - \left[\widehat{\pi} - (1-\delta)F\right] \\
&+ \sum_{t=T+1}^{\infty} \delta^{t-T}\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right] \prod_{j=T+1}^{t-1}\left[1 - \phi\left(\overline{P}^j + \varepsilon, \overline{P}^{j-1} + \varepsilon\right)\right] \times \\
&\left[\pi\left(\overline{P}^t + \varepsilon\right) - \overline{\Delta}^t \gamma x\left(\overline{P}^t + \varepsilon\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] + \left[\widehat{\pi}/(1-\delta) - F\right],
\end{aligned}$$

where $\overline{\Delta}^t \equiv \delta \sum_{\tau=t}^{\infty} (\delta\beta)^{\tau-t} \phi\left(\overline{P}^\tau + \varepsilon, \overline{P}^{\tau-1} + \varepsilon\right) \prod_{j=t}^{\tau-1}\left[1 - \phi\left(\overline{P}^j + \varepsilon, \overline{P}^{j-1} + \varepsilon\right)\right]$.

This payoff is continuous in ε and equals the payoff from $\left\{\overline{P}\right\}_{t=T}^{\infty}$ when $\varepsilon = 0$. Optimality requires that if the derivative of the payoff with respect to ε is defined then it equals 0 at $\varepsilon = 0$. Prior to taking the derivative, recall that

$$\phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right) = \widehat{\phi}\left(\left(\overline{P}^T + \varepsilon - \overline{P}^{T-1}\right) g\left(\overline{P}^{T-1}\right)\right),$$

$$\phi\left(\overline{P}^t + \varepsilon, \overline{P}^{t-1} + \varepsilon\right) = \widehat{\phi}\left(\left(\overline{P}^t - \overline{P}^{t-1}\right) g\left(\overline{P}^{t-1} + \varepsilon\right)\right),\ t > T$$

When the derivative of ϕ is taken, it'll be replaced with its alternative representation of $\widehat{\phi}$ for purposes of the analysis.

Taking the derivative of the payoff with respect to ε :

$$\begin{aligned}
&\pi'\left(\overline{P}^T + \varepsilon\right) - \left(1 - \beta\overline{\Delta}^{T+1}\right)\delta\widehat{\phi}'\left(\left(\overline{P}^T + \varepsilon - \overline{P}^{T-1}\right) g\left(\overline{P}^{T-1}\right)\right) g\left(\overline{P}^{T-1}\right) \times \\
&\left[\gamma x\left(\overline{P}^T + \varepsilon\right) + \beta X^{T-1}\right] \\
&-\delta\beta\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right]\left[\gamma x\left(\overline{P}^T + \varepsilon\right) + \beta X^{T-1}\right]\left(\partial\overline{\Delta}^{T+1}/\partial\varepsilon\right) \\
&- \left\{\delta\phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right) + \delta\beta\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right]\overline{\Delta}^{T+1}\right\}\gamma x'\left(\overline{P}^T + \varepsilon\right) \\
&- \sum_{t=T+1}^{\infty} \delta^{t-T}\widehat{\phi}'\left(\left(\overline{P}^T + \varepsilon - \overline{P}^{T-1}\right) g\left(\overline{P}^{T-1}\right)\right) g\left(\overline{P}^{T-1}\right) \prod_{j=T+1}^{t-1}\left[1 - \phi\left(\overline{P}^j + \varepsilon, \overline{P}^{j-1} + \varepsilon\right)\right] \times \\
&\left[\pi\left(\overline{P}^t + \varepsilon\right) - \overline{\Delta}^t \gamma x\left(\overline{P}^t + \varepsilon\right) - \left(\widehat{\pi} - (1-\delta)F\right)\right] \\
&- \sum_{t=T+1}^{\infty} \delta^{t-T}\left[1 - \phi\left(\overline{P}^T + \varepsilon, \overline{P}^{T-1}\right)\right] \times \\
&\sum_{j=T+1}^{t-1} \widehat{\phi}'\left(\left(\overline{P}^j - \overline{P}^{j-1}\right) g\left(\overline{P}^{j-1} + \varepsilon\right)\right)\left(\overline{P}^j - \overline{P}^{j-1}\right) g'\left(\overline{P}^{j-1} + \varepsilon\right) \times
\end{aligned}$$

$$\prod_{\substack{k=T+1\\k\neq j}}^{t-1}\left[1-\phi\left(\overline{P}^{k}+\varepsilon,\overline{P}^{k-1}+\varepsilon\right)\right]\left[\pi\left(\overline{P}^{t}+\varepsilon\right)-\overline{\Delta}^{t}\gamma x\left(\overline{P}^{t}+\varepsilon\right)-\left(\widehat{\pi}-(1-\delta)F\right)\right]$$

$$-\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\prod_{j=T+1}^{t-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]\gamma x\left(\overline{P}^{t}+\varepsilon\right)\left(\partial\overline{\Delta}^{t}/\partial\varepsilon\right)$$

$$+\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\phi\left(\overline{P}^{T}+\varepsilon,\overline{P}^{T-1}\right)\right]\prod_{j=T+1}^{t-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]\times$$

$$\left[\pi'\left(\overline{P}^{t}+\varepsilon\right)-\overline{\Delta}^{t}\gamma x'\left(\overline{P}^{t}+\varepsilon\right)\right].$$

$$\begin{aligned}\text{where }\frac{\partial\overline{\Delta}^{t}}{\partial\varepsilon} &= \delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\widehat{\phi}'\left(\left(\overline{P}^{\tau}-\overline{P}^{\tau-1}\right)g\left(\overline{P}^{\tau-1}+\varepsilon\right)\right)\left(\overline{P}^{\tau}-\overline{P}^{\tau-1}\right)g'\left(\overline{P}^{\tau-1}+\varepsilon\right)\times\\ &\quad\prod_{j=t}^{\tau-1}\left[1-\phi\left(\overline{P}^{j}+\varepsilon,\overline{P}^{j-1}+\varepsilon\right)\right]-\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\phi\left(\overline{P}^{\tau}+\varepsilon,\overline{P}^{\tau-1}+\varepsilon\right)\times\\ &\quad\sum_{j=t}^{\tau-1}\widehat{\phi}'\left(\left(\overline{P}^{j}-\overline{P}^{j-1}\right)g\left(\overline{P}^{j-1}+\varepsilon\right)\right)\left(\overline{P}^{\tau}-\overline{P}^{\tau-1}\right)g'\left(\overline{P}^{j-1}+\varepsilon\right)\times\\ &\quad\prod_{\substack{k=t\\k\neq j}}^{\tau-1}\left[1-\phi\left(\overline{P}^{k}+\varepsilon,\overline{P}^{k-1}+\varepsilon\right)\right].\end{aligned}$$

Optimality requires that this derivative (if defined) equals zero at $\varepsilon=0$, $\forall T$. Consider this derivative, evaluated at $\varepsilon=0$, as $T\to\infty$. Since $\lim_{T\to\infty}\overline{P}^{T}=\overline{P}$ then

$$\begin{aligned}\lim_{T\to\infty}\widehat{\phi}'\left(\left(\overline{P}^{T}-\overline{P}^{T-1}\right)g\left(\overline{P}^{T-1}\right)\right)g\left(\overline{P}^{T-1}\right) &= \widehat{\phi}'(0)=0\\ \lim_{T\to\infty}\widetilde{\phi}'\left(\left(\overline{P}^{t}-\overline{P}^{t-1}\right)g\left(\overline{P}^{t-1}\right)\right)g'\left(\overline{P}^{t-1}\right) &= \widetilde{\phi}'(0)=0,\ t>T.\end{aligned}$$

Since $\widetilde{\phi}'(0)$ is defined by A9, then the above derivative of the payoff function is defined. Thus, as $T\to\infty$, all of the expressions with $\widehat{\phi}'$ equal zero as do $\partial\overline{\Delta}^{T}/\partial\varepsilon$ and $\partial\overline{\Delta}^{t}/\partial\varepsilon$. This leaves:

$$\begin{aligned}&\pi'\left(\overline{P}\right)-\left\{\delta\widehat{\phi}(0)+\delta\beta\left[1-\widehat{\phi}(0)\right]\overline{\Delta}\right\}\gamma x'\left(\overline{P}\right)\\ &\quad+\sum_{t=T+1}^{\infty}\delta^{t-T}\left[1-\widehat{\phi}(0)\right]\prod_{j=T+1}^{t-1}\left[1-\widehat{\phi}(0)\right]\left[\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma x'\left(\overline{P}\right)\right]\\ =&\sum_{t=T}^{\infty}\delta^{t-T}\left[1-\widehat{\phi}(0)\right]^{t-T}\left[\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma x'\left(\overline{P}\right)\right]\\ =&\frac{\pi'\left(\overline{P}\right)-\overline{\Delta}\gamma x'\left(\overline{P}\right)}{1-\delta\left(1-\widehat{\phi}(0)\right)},\end{aligned}$$

$$\text{where }\overline{\Delta}\equiv\delta\sum_{\tau=t}^{\infty}(\delta\beta)^{\tau-t}\widehat{\phi}(0)\left[1-\widehat{\phi}(0)\right]^{\tau-t}=\frac{\delta\widehat{\phi}(0)}{1-\delta\beta\left(1-\widehat{\phi}(0)\right)}.$$

Optimality then requires that $\pi'\left(\overline{P}\right) - \overline{\Delta}\gamma x'\left(\overline{P}\right) = 0$ which, by A11, implies $\overline{P} = P^*$. This completes the proof of Theorem 3. ■

Appendix B

Key to our analysis is a useful representation of a firm's payoff. To save on notation, let $\phi^t \equiv \phi\left(P^t, P^{t-1}\right)$ denote the probability of detection in period t, as of the start of period t. Suppose collusion is infinitely-lived (subject to detection interrupting it) and the collusive price path is $\{P^t\}_{t=1}^{\infty}$. The payoff as of period t can then be represented as:

$$\begin{aligned}
&\pi\left(P^t\right) + \delta\phi^t\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta X^{t-1} - \gamma x\left(P^t\right) - F\right] + \delta\left(1-\phi^t\right)\pi\left(P^{t+1}\right) \\
&+\delta^2\left(1-\phi^t\right)\phi^{t+1}\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^2 X^{t-1} - \beta\gamma x\left(P^t\right) - \gamma x\left(P^{t+1}\right) - F\right] \\
&+\delta^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\pi\left(P^{t+2}\right) \\
&+\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2} \times \\
&\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - \beta^3 X^{t-1} - \beta^2\gamma x\left(P^t\right) - \beta\gamma x\left(P^{t+1}\right) - \gamma x\left(P^{t+2}\right) - F\right] + \cdots
\end{aligned}$$

A firm earns $\pi\left(P^t\right)$ in the current period. With probability ϕ^t, detection occurs which results in $\widehat{\pi}$ in all future periods and a penalty of $\beta X^{t-1} + \gamma x\left(P^t\right) + F$. With probability $1-\phi^t$, detection does not occur so $\pi\left(P^{t+1}\right)$ is earned in period $t+1$ and so forth. This expression can be re-arranged to:

$$\begin{aligned}
&\{\pi\left(P^t\right) - \gamma x\left(P^t\right)\delta\left[\phi^t + \delta\beta\left(1-\phi^t\right)\phi^{t+1} + \left(\delta\beta\right)^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2} + \cdots\right] \\
&+\delta\left(1-\phi^t\right)\pi\left(P^{t+1}\right) \\
&-\delta\left(1-\phi^t\right)\gamma x\left(P^{t+1}\right)\delta\left[\phi^{t+1} + \delta\beta\left(1-\phi^{t+1}\right)\phi^{t+2} + \left(\delta\beta\right)^2\left(1-\phi^{t+1}\right)\left(1-\phi^{t+2}\right)\phi^{t+3} + \cdots\right] \\
&+\cdots\} + \left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\left[\delta\phi^t + \delta^2\left(1-\phi^t\right)\phi^{t+1} + \delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2} + \cdots\right] \\
&-\beta X^{t-1}\delta\left[\phi^t + \delta\beta\left(1-\phi^t\right)\phi^{t+1} + \left(\delta\beta\right)^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2} + \cdots\right].
\end{aligned}$$

Let

$$\Delta^t \equiv \delta\sum_{\tau=t}^{\infty}\left(\delta\beta\right)^{\tau-t}\phi^{\tau}\prod_{j=t}^{\tau-1}\left[1-\phi^j\right],$$

where the convention is adopted that $\prod_{j=t}^{t-1}\left[1-\phi^j\right] = 1$. The above expression is then:

$$\begin{aligned}
&\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)\left[\pi\left(P^{\tau}\right) - \gamma x\left(P^{\tau}\right)\Delta^{\tau}\right] \qquad (24) \\
&+\left[\left(\widehat{\pi}/\left(1-\delta\right)\right) - F\right]\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^{\tau}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right) - \beta X^{t-1}\Delta^t.
\end{aligned}$$

The collusive payoff is represented as the stream of profit net of the expected present value of damages, $\pi\left(P^{\tau}\right) - \gamma x\left(P^{\tau}\right)\Delta^{\tau}$, less the expected present value of the fine, $\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^{\tau}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)F$,

less the expected present value of inherited damages, $\beta X^{t-1}\Delta^t$, plus the value from not colluding, $(\widehat{\pi}/(1-\delta))\,\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^{\tau}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)$.

Let us manipulate the term $[(\widehat{\pi}/(1-\delta))-F]\,\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^{\tau}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)$:

$$
\begin{aligned}
& [(\widehat{\pi}/(1-\delta))-F]\,\delta\sum_{\tau=t}^{\infty}\delta^{\tau-t}\phi^{\tau}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)\\
= \;& [(\widehat{\pi}/(1-\delta))-F]\left\{\delta\phi^t+\delta^2\left(1-\phi^t\right)\phi^{t+1}+\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}+\cdots\right\}\\
= \;& [\widehat{\pi}-(1-\delta)F]\left\{\delta\phi^t\sum_{\tau=0}^{\infty}\delta^{\tau}+\delta^2\left(1-\phi^t\right)\phi^{t+1}\sum_{\tau=0}^{\infty}\delta^{\tau}\right.\\
& \left.+\,\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\sum_{\tau=0}^{\infty}\delta^{\tau}+\cdots\right\}\\
= \;& [\widehat{\pi}-(1-\delta)F]\left\{\delta\phi^t+\delta^2\left[\phi^t+\left(1-\phi^t\right)\phi^{t+1}\right]+\right.\\
& \left.\delta^3\left[\phi^t+\left(1-\phi^t\right)\phi^{t+1}+\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\right]+\cdots\right\}\\
= \;& [(\widehat{\pi}/(1-\delta))-F]-[\widehat{\pi}-(1-\delta)F]\left[\left(1+\delta+\delta^2+\cdots\right)-\delta\phi^t\right.\\
& -\delta^2\left(\phi^t+\left(1-\phi^t\right)\phi^{t+1}\right)\\
& \left.-\delta^3\left(\phi^t+\left(1-\phi^t\right)\phi^{t+1}+\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\right)-\cdots\right]\\
= \;& [(\widehat{\pi}/(1-\delta))-F]-[\widehat{\pi}-(1-\delta)F]\times\\
& \left\{1+\delta\left(1-\phi^t\right)+\delta^2\left[1-\phi^t-\left(1-\phi^t\right)\phi^{t+1}\right]\right.\\
& \left.+\delta^3\left[1-\phi^t-\left(1-\phi^t\right)\phi^{t+1}-\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\phi^{t+2}\right]+\cdots\right\}\\
= \;& [(\widehat{\pi}/(1-\delta))-F]-[\widehat{\pi}-(1-\delta)F]\times\\
& \left\{1+\delta\left(1-\phi^t\right)+\delta^2\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\right.\\
& \left.+\delta^3\left(1-\phi^t\right)\left(1-\phi^{t+1}\right)\left(1-\phi^{t+2}\right)+\cdots\right\}\\
= \;& [(\widehat{\pi}/(1-\delta))-F]-[\widehat{\pi}-(1-\delta)F]\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right).
\end{aligned}
$$

Substituting this expression into (24):

$$
\begin{aligned}
& \sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pi_{j=t}^{\tau-1}\left(1-\phi^j\right)\left\{[\pi\left(P^{\tau}\right)-\gamma x\left(P^{\tau}\right)\Delta^{\tau}]-[\widehat{\pi}-(1-\delta)F]\right\} \qquad (25)\\
& -\beta X^{t-1}\Delta^t+[(\widehat{\pi}/(1-\delta))-F].
\end{aligned}
$$

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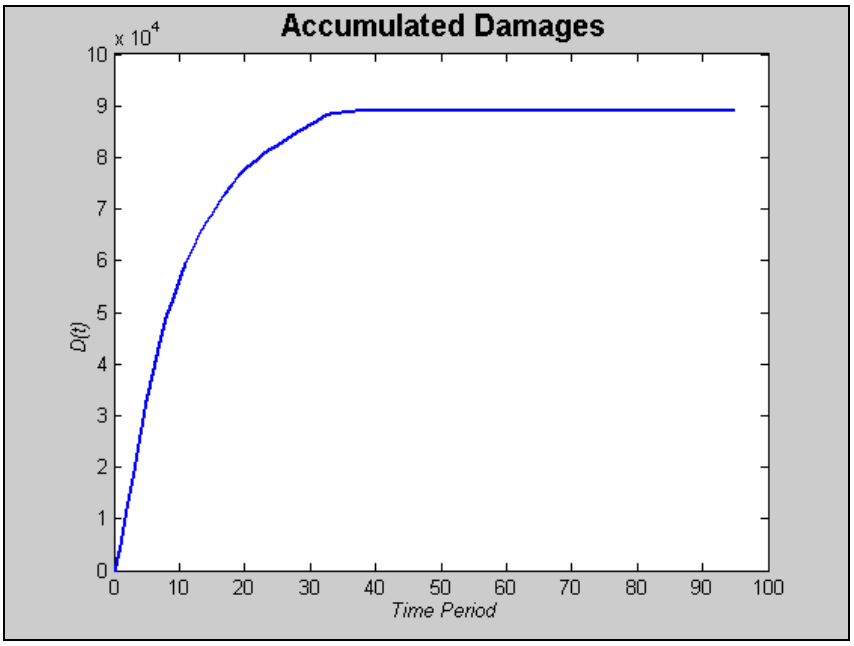
Figure 1


Accumulated Damages
x 10^4
10
9
8
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5
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3
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10
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40
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60
70
80
90
100
Time Period

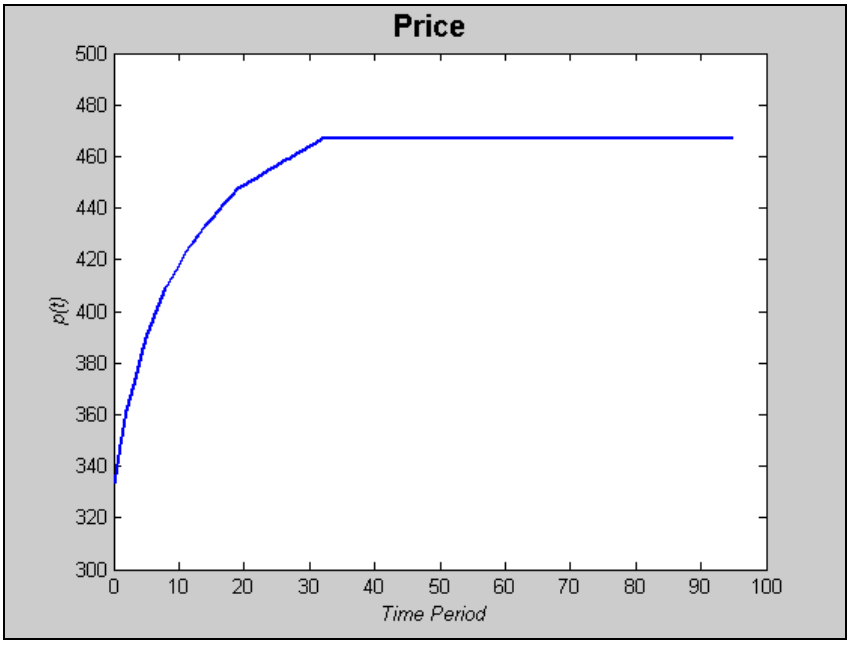
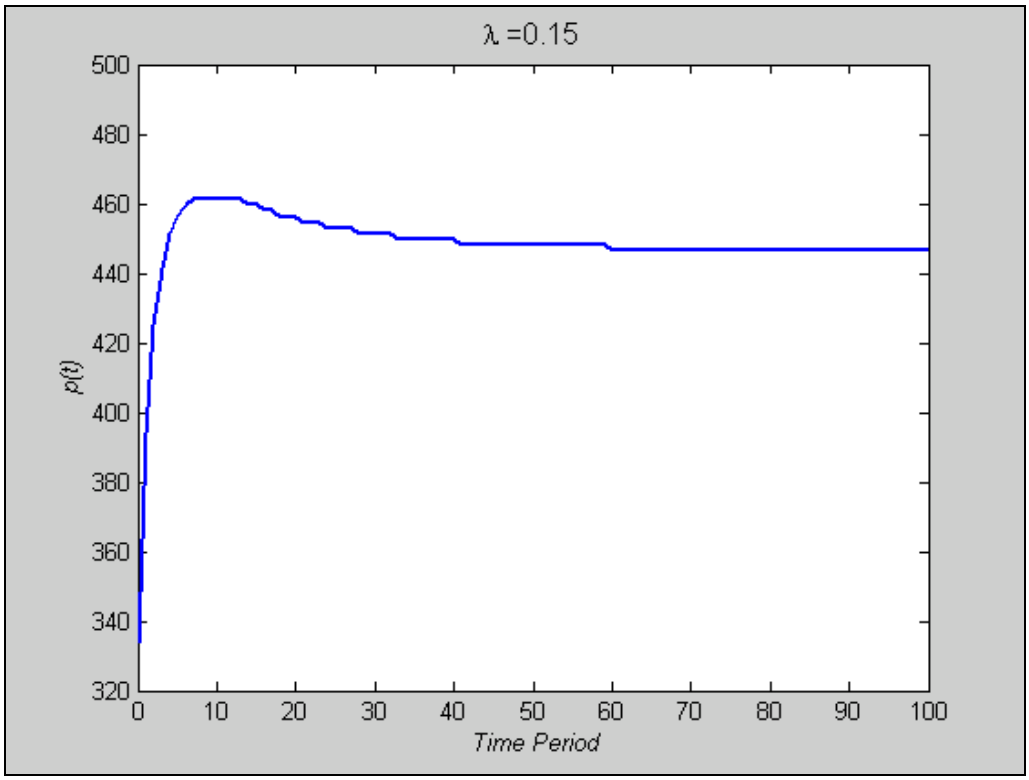

Price
500
480
460
440
420
400
380
360
340
320
300
p(t)
0
10
20
30
40
50
60
70
80
90
100
Time Period

Figure 2


λ =0.15
500
480
460
440
420
400
380
360
340
320
p(t)
0
10
20
30
40
50
60
70
80
90
100
Time Period