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Exact Arbitrage, Well-Diversified Portfolios and Asset Pricing in Large Markets*

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Abstract: For a market with an atomless continuum of assets, we formulate the intuitive idea of a "well-diversified" portfolio, and present a notion of "exact arbitrage", strictly weaker than the more conventional notion of "asymptotic arbitrage", and necessary and sufficient for the validity of an APT pricing formula. Our formula involves "essential" risk, one based on a specific index portfolio constructed from factors and factor loadings that are endogenously extracted to satisfy an optimality property involving a finite number of factors. We illustrate how our results can be translated to markets with a large but finite number of assets.*Journal of Economic Literature* Classification Numbers: G12, C60.

Key Words: exact arbitrage, asymptotic arbitrage, exact law of large numbers, well-diversified portfolio, essential risk, arbitrage pricing theory, Loeb measure space.

Running Title: Exact Arbitrage Pricing Theory

1 Introduction

Ross' [35, 36] arbitrage pricing theory (APT) is an attempt to formalize the raw intuition that "well-diversified portfolios exhibit no idiosyncratic risk", and thereby to derive an APT asset-pricing formula for markets which do not permit any possibility of gains from arbitrage. The claim is that the market does not reward gains from *naive diversification* in an arbitrage-free environment, and therefore the expected rate of return to a particular asset is approximately linearly related to its factor loadings, which is to say, determined solely by factors formalizing *systematic* risk. This intuition concerning naive diversification is based simply on portfolio size and correspondingly draws on a version of the classical law of large numbers. It is therefore important to note that it differs from that of the capital-asset-pricing model (CAPM) of Sharpe [40] and Lintner [28] where the distinction between *non-diversifiable* and *diversifiable* risks is based on mean-variance efficiency, and thereby on the *efficient diversification* of a portfolio.[1]

However, despite several attempts,[2] the intuitive notion of a "well-diversified" portfolio has resisted a rigorous and precise treatment. In fact, Chamberlain-Rothschild [7, p.1282] flatly state, "Ross' heuristics cannot be made rigorous."[3] In this paper, we offer an APT theory of considerable scope and power that not only dispels these, and other, misgivings, but also allows us to uncover concepts that have so far been missed in the literature. In particular, our theoretical framework allows us to present a notion of no-arbitrage gains that is directly suggested by the popular aphorism, *"there are no profitable opportunities without cost or risk"*. As such, it is both appealing and extremely simple: a portfolio with zero cost and zero risk has a zero return. Such an assumption is strictly weaker than the more elaborate asymptotic no-arbitrage assumption common in the conventional APT literature,[4] and unlike it, both necessary and sufficient for an APT asset-pricing formula to hold.

Our model is based on a "large" number of asset names formalized as an atomless measure space, and the random returns of the assets are described by a real-valued stochastic process indexed by elements of such a space. The measure-theoretic structure is used to formulate the intuitive notion of "negligibility", and then by defining unsystematic risk in terms of its effect on a "negligible" corner of the financial market, to capture in a precise way the common

[1] See [33, pp. 173-197] for a discussion of naive and efficient diversification. We note here that there is no uniform terminology in the literature. For example, the terms *non-diversifiable* risk and *diversifiable* risk used here for the CAPM are also called *systematic* risk and *unsystematic* risk in [40]. On the other hand, *systematic* risk and *unsystematic* risk used here for the APT model are also referred to as *non-diversifiable* and *diversifiable* risk in, for example, [38, pp. 116-120] and [37].

[2] See [27]; we comment further on these attempts in the sequel.

[3] After presenting the intuitive definition of an unsystematic risk, Ross et al. [37, p. 294]) note that "we may not be able to define a systematic risk and an unsystematic risk exactly, but we know them when we see them." Also see Al-Najjar's [1, p. 248-249] comments on how difficult it is "to formalize (let alone prove) within the sequence model" the basic intuition.

[4] As, for example, in [35], [36], [6],[7],[10],[18], [19]. In the sequel, we shall no longer refer to these papers when we use the phrase "conventional APT literature" but simply to this footnote.

intuition underlying the distinction between systematic and unsystematic risks. Specifically, we use the results in [41, 42], based on a particular class of measure spaces due to Loeb [29], to model probabilistic phenomenon involving a "large" number of random variables in situations where there is no natural topology on the set indexing these random variables. A key feature of this framework is that well-known measurability issues related to the complete cancellation of idiosyncratic risks are automatically resolved, and various versions of the *exact* law of large numbers can be proved. In our particular context, it means that unsystematic risks can be completely eliminated under naive diversification. This naturally leads to the formulation of the no-arbitrage condition in a more concrete and transparent way, which is then shown to be equivalent to the APT valuation formula under the simple projection method in the theory of Hilbert spaces. The underlying biorthogonal representation theorem also allows an endogenous and optimal extraction of the factors by identifying the eigenfunctions of the associated autocorrelation function of the process.

Our theoretical framework enables us to identify an index portfolio I_0 in terms of which systematic risks can be further decomposed into what we shall call here *essential* and *inessential* risks. Whereas unsystematic risk can be eliminated by naive diversification, inessential risk is correlated with a non-negligible portion of the market and can only be eliminated by efficient diversification. We thus obtain a valuation formula in which the expected return of an asset t is exactly linearly dependent on its beta, β_t, but now recalculated to be the covariance of the asset's random return with that of the index portfolio I_0. This is an I_0-based Beta model of the rates of return,[5] and it sharply brings out the fact that in spite of many factor risks, the market only rewards a risk which is essential and which can never be eliminated through either naive or efficient diversification. The novel feature here is that we distinguish three types of risk (unsystematic, essential and inessential risks) explicitly and study the exact nature of two different types of diversification (naive and efficient) in one model. This is in contrast to the previous asset pricing literature where only two of these three types of risks are identified, and handled, one pair at a time, and in different settings.

The idealized limit model thus uncovers phenomena obscured in the discrete case, and also not readily apparent in the large but finite case. It is only after the identification of the exact no-arbitrage assumption in the ideal case that we know what approximations to look for in the large but finite case. In such a setting, by necessity, each asset occupies a non-negligible portion of the market, and the exact no-arbitrage assumption is expressed in an asymptotic form that is different from the usual asymptotic no-arbitrage assumption in the literature. It is an additional strength of the idealized setting that it allows a translation of the intuitive and transparent regularities of the idealized case into (of necessity, rather complicated epsilon-cluttered) statements for the large but finite case. In classical treatments of the APT, as in

[5] See Duffie's text [12, Chapter 1] for details and additional references.

[6],[7], factor structures are not refined enough,[6] well-diversified portfolios can only be defined as limits of the returns of finite portfolios, and by requiring the sum of an infinite series to be finite, the approximate results do not give equal treatment to each asset. It is because of this that one can understand why the concepts singled out here as important and relevant, both in the idealized and non-idealized cases,[7] have not been identified in earlier work.

The remainder of the paper is organized in two substantive sections, followed by a conclusion that highlights its principal contribution, and an Appendix that collects the technicalities of one proposition. In Section 2, we present the principal results and the basic contours of the theory, and in Section 3, explain how it impacts on, and clarifies, concepts that are current in the literature.

2 Exact Arbitrage, Risk Analysis and Asset Pricing

It is well-known that there are measurability problems associated with the sample functions as well as with the relevant joint process when idiosyncratic risks are considered in the setting of a Lebesgue continuum of assets; see Doob [8, 9] and Judd [21]. A key feature in [41, 42] is the use of the Loeb measure framework to resolve these measurability issues automatically.

We first recall some basic definitions from [20] and [23]. Let $(T, \mathcal{T}, \lambda)$ be the Loeb counting probability space on a hyperfinite set T, to be used as the index set of assets.[8] We work with another atomless Loeb measure space $(\Omega, \mathcal{A}, P)$ as the sample space, a space that formalizes all possible uncertain social or natural states relevant to the asset market. The usual product space is denoted by $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$. The crucial point is that even though this product space is not large enough for the study of idiosyncratic risks, there is another product space $(T \times \Omega, \mathcal{T} \otimes^L \mathcal{A}, \lambda \otimes^L P)$, the Loeb product space, that extends the usual product, retains the Fubini property and is rich enough for the study of idiosyncratic risks (see [42]). As is conventional, we shall refer to a measurable function of two variables as a process. Given a process g on the Loeb product space, for each $t \in T$, and each $\omega \in \Omega$, g_t denotes the function $g(t, \cdot)$ on Ω and g_ω denotes the function $g(\cdot, \omega)$ on T. The functions g_t are usually called the random variables of the process g, while the g_ω are referred to as the sample functions of the process.[9] Since the measure $\lambda \otimes^L P$ is an extension of $\lambda \otimes P$ on the usual product σ-algebra

[6] The factor loadings in the hyperfinite factor model in [42] are also orthonormal; see Theorem A below.

[7] We hope that the reader will appreciate, especially after reading Section 3.2, that the scientific content of a result for the idealized and the non-idealized (large but finite) cases is identical.

[8] Note that $(T, \mathcal{T}, \lambda)$ is an atomless measure space constructed from an internal counting probability space on $(T, \bar{\mathcal{T}}, \bar{\lambda})$. As is by now well-understood in the economics literature, Loeb measure spaces, even though constituted by nonstandard entities, are standard measure spaces in the specific sense that any result proved for an abstract measure space applies to them; for details, see [2] and its references. For a mathematical introduction to nonstandard analysis and to applications, see, for example, [20]. It is of course not necessary to use the counting measure on T; it is simply a natural one given our interest in translating the (idealized) limit results to a sequence of large but finite markets.

[9] Note that the measurability of g_t and g_ω is a simple consequence of a Fubini type result for Loeb product

$\mathcal{T} \otimes \mathcal{A}$ to the larger product σ-algebra $\mathcal{T} \otimes^L \mathcal{A}$, for notational simplicity, we replace $\lambda \otimes^L P$ by $\lambda \otimes P$ in the sequel. We emphasize that we always work with the larger product σ-algebra $\mathcal{T} \otimes^L \mathcal{A}$ since $\mathcal{T} \otimes \mathcal{A}$ is "too small" to be endowed with any non-trivial independent risks (see Proposition 1.1 in [43]).

We shall model the financial market[10] by a real-valued $\mathcal{T} \otimes^L \mathcal{A}$-measurable function x on $T \times \Omega$, and interpret the real-valued random variable x_t defined on $(\Omega, \mathcal{A}, P)$ as the one-period random return to an asset t in T. In order to use the notion of the variance of the return to any asset, we shall assume that the asset return process x has a finite second moment, and therefore belongs to the Hilbert space $\mathcal{L}^2(\lambda \otimes P)$ of real-valued square integrable functions on the Loeb product space.[11] Thus the square of the norm of x is given by the inner product

$$(x, x) = \int \int_{T \times \Omega} x^2(t, \omega) d\lambda \otimes P(t, \omega) \ < \ \infty. \tag{1}$$

Let μ be the mean function of the random variables embodied in the process x of asset returns,[12] which is to say that $\mu(t) = \int_\Omega x(t, \omega) dP(\omega)$ is the expected return of asset $t \in T$. By the Cauchy-Schwarz inequality, it is clear that

$$\int_T \mu^2(t) d\lambda \leq \int \int_{T \times \Omega} x^2(t, \omega) d\lambda \otimes P(t, \omega) < \infty, \tag{2}$$

and hence μ is λ-square integrable and belongs to the Hilbert space $\mathcal{L}^2(\lambda)$. The centered process f, defined by $f(t, \omega) = x(t, \omega) - \mu(t)$, embodies the unexpected or the net random return of all the assets, and is also $\lambda \otimes P$-square integrable.[13]

A portfolio is simply a function listing the amounts held of each asset. Since short sales are allowed, this function can take negative values. The cost of each asset is assumed to be unity, and hence the cost of a particular portfolio is simply its integral with respect to λ. Since we are interested in the mean and variance of the return realized from a portfolio, we shall assume it to be a square integrable function. The random return from a particular portfolio then depends on the random return, and the amounts held in the portfolio, of each asset $t \in T$. Formally,

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measures (also referred to as Keisler's Fubini theorem); see [23].

[10] In the sequel, we shall also refer to this simply as a "market".

[11] As exposited, for example, in [39, Chapter 4]. It is worth noting that in the context of the subject matter of this paper, projection maps on Hilbert spaces have played a fundamental role in [6], [7], [19], [34] and [27].

[12] It is clear that x is also $\lambda \otimes P$-integrable. An appeal to the Fubini type theorem for Loeb measures as shown by Keisler, then guarantees that μ is a Loeb integrable function on $(T, \mathcal{T}, \lambda)$. In the sequel, $\mu(t)$ will also be denoted by μ_t. One can understand a Fubini type result on iterated integrals in the hyperfinite Loeb measure setting as the simple observation that hyperfinite sums can be exchanged. Hereafter, when the need arises, we simply change the order of integrals without an explicit statement on the application of any Fubini type results.

[13] As noted in [37, p. 293], "the unanticipated part of the return, that portion resulting from surprises, is the true risk of any investment. After all, if we had already got what we had expected, there would be no risk and no uncertainty."

Definition 1 *A portfolio is a square integrable function p on $(T, \mathcal{T}, \lambda)$. The cost $C(p)$ of a portfolio p is given by $(p, 1) = \int_T p(t)d\lambda(t)$. The random return of the portfolio p is given by $\mathcal{R}_p(\omega) = (p, x_\omega) = \int_T p(t)x(t, \omega)d\lambda(t)$. The mean (or the expected return) $E(p)$ and the variance $V(p)$ of the portfolio p are the mean and the variance of the random return $\mathcal{R}_p$ respectively.*

Heuristically, $d\lambda(t)$ is interpreted as an infinitesimal amount of an asset t and can be regarded as a small accounting unit in some sense. Thus, in the portfolio p, $p(t)d\lambda(t)$ is the amount, and $p(t)x(t, \omega)d\lambda(t)$ is the return, of shares of asset $t \in T$. For any two assets, s and t in T, $p(s)$ and $p(t)$ measure their relative amounts in the portfolio p. Since these terms are integrable as a function of t over an atomless measure space, the amount invested in, and the return pertaining to, any asset is infinitesimal, and therefore any portfolio is well-diversified automatically.

We now turn to the decomposition of an asset's return into systematic and unsystematic (synonymously, factor and idiosyncratic) risks.

2.1 Systematic and Unsystematic Risks: A Bi-variate Decomposition

We begin by noting the interesting fact that the completion of the standard product measure space[14] $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$, corresponding to the standard measure spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$, is always strictly contained in the Loeb product space $(T \times \Omega, \mathcal{T} \otimes^L \mathcal{A}, \lambda \otimes P)$.[15] For simplicity, let $\mathcal{U}$ denote the product σ-algebra $\mathcal{T} \otimes \mathcal{A}$. For an integrable real-valued process g on the Loeb product space, let $E(g|_{\mathcal{U}})$ denote the conditional expectation[16] of g with respect to $\mathcal{U}$. This conditional expectation is a key operation,[17] introduced in [41], and used here to formalize the ensemble of systematic risks and unsystematic risks, and thereby to model uncertainty from both the macroscopic and microscopic points of view. It makes rigorous the pervasive attempts in the economic literature that use a discrete or continuous parameter process with low intercorrelation to model individual uncertainty, and then to invoke the law of large numbers to remove this individual uncertainty.[18]

We now collect for the reader's convenience relevant results from [42, Corollary 4.8]), [41, Theorems 1-3], as a portmanteau theorem.

Theorem A *Let f be a real-valued square integrable centered process on the Loeb product space $(T \times \Omega, \mathcal{T} \otimes^L \mathcal{A}, \lambda \otimes P)$. Then f has the following expression: for $\lambda \otimes P$-almost all (t, ω) in*

[14] For product measures, and for integration on such measures, see, for example, [30, Chapter VIII] or [39, Chapter 7].

[15] See [42, Proposition 6.6].

[16] See [31, Chapter VIII] for details as to conditional expectations.

[17] Note that such an operation involves both a product σ-algebra and a natural but significant extension of it: $\mathcal{U}$ and the Loeb product algebra $\mathcal{T} \otimes^L \mathcal{A}$. As such, it has no natural counterpart in standard mathematical practice or in nonstandard mathematics using only internal entities; see [41, 42] for details.

[18] Of course, as emphasized in the introduction, and also in Part 3 below, the continuous case does not allow a viable law of large numbers, and the discrete case results in only an approximate removal.

$T \times \Omega$,

$$f(t,\omega) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega) + e(t,\omega),$$

with properties:

(i) $\lambda_n, 1 \leq n < \infty$ *is a decreasing sequence of positive numbers; the collection* $\{\psi_n : 1 \leq n < \infty\}$ *is orthonormal; and* $\{\varphi_n : 1 \leq n < \infty\}$ *is a collection of orthonormal and centered random variables.*

(ii) $E(f|_{\mathcal{U}})(t,\omega) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega)$ *and* $E(e|_{\mathcal{U}}) = 0$.

(iii) The random variables e_t *are almost surely orthogonal, which is to say that for* $\lambda \otimes \lambda$*-almost all* $(t_1, t_2) \in T \times T$,

$$\int_{\Omega} e_{t_1}(\omega) e_{t_2}(\omega) dP(\omega) = 0.$$

(iv) If p *is a square integrable real-valued function on* $(T, \mathcal{T}, \lambda)$*, then for* P*-almost all* $\omega \in \Omega$,

$$\int_T p(t) e_\omega(t) d\lambda(t) = 0 \text{ and } \int_T p(t) f(t,\omega) d\lambda(t) = \sum_{n=1}^{\infty} \lambda_n \left(\int_T p(t)\psi_n(t) d\lambda(t) \right) \varphi_n(\omega).$$

(v) If α *is a square integrable random variable on* $(\Omega, \mathcal{A}, P)$*, then for* λ*-almost all* $t \in T$*, it is orthogonal to* e_t*, and*

$$\int_{\Omega} \alpha(\omega) f(t,\omega) dP(\omega) = \sum_{n=1}^{\infty} \lambda_n \left(\int_{\Omega} \alpha(\omega)\varphi_n(\omega) dP(\omega) \right) \psi_n(t).$$

The structural result in Theorem A can be seen as a hyperfinite version of the classical factor model with a finite population: the centered random variables φ_n as factors, the corresponding functions ψ_n as factor loadings, and the decreasing sequence of numbers λ_n as scaling constants, with the size of λ_n measures the role of the factor φ_n in understanding the correlational structure of f. It is worth emphasizing that the factors are endogenously derived by virtue of the fact that the associated autocorrelation function of the process f, $R(t_1, t_2) = \int_\Omega f(t_1,\omega) f(t_2,\omega) dP$, by serving as a kernel, defines an integral operator K on the space $\mathcal{L}^2(\lambda)$ of square integrable functions on $(T, \mathcal{T}, \lambda)$. That is, $K(h)(t_1) = \int_T R(t_1, t_2) h(t_2) d\lambda(t_2)$ for $h \in \mathcal{L}^2(\lambda)$. It is easily checked that λ_n^2 is in fact the n-th positive eigenvalue of the operator K with eigenfunction ψ_n, with all the eigenvalues listed according to reverse order and repeated up to their corresponding multiplicities.[19] It is also clear that $\varphi_n(\omega) = (1/\lambda_n) \int_T f_\omega(t)\psi_n(t) d\lambda$.

Before commenting on the five claims of Theorem A, let us refer to a *risk* as any centered random variable with a finite variance defined on the sample space Ω. We shall measure a *level*

[19] Note that if there are only m positive eigenvalues, then the infinite sum in Theorem A should be read in the sequel as a finite sum of m terms.

of risk by the variance of the risk. Thus, the risk in asset t is simply the net random return f_t of asset t, and the process f is the ensemble of all the risks present in the financial market. The following definition then encapsulates the intuitive notion of an unsystematic risk.

Definition 2 *A centered random variable α on the sample space Ω is said to be an unsystematic risk for a financial market x if α has finite variance and is uncorrelated to x_t for λ-almost all $t \in T$.*

We can now see that Theorem A presents, simply by virtue of the fact that the underlying process is square integrable, a bi-variate decomposition of an asset's unexpected rate of return into endogenously derived categories of unsystematic and systematic risks. Conditions (i) and (ii) say that the conditional expectation $E(f|_{\mathcal{U}})$ has a biorthogonal expansion in which both the random variables φ_n and the functions ψ_n are orthogonal among themselves.[20]

Condition (iii) pertains to the use of the process e as an expression of the ensemble of unsystematic risks in the financial market, and makes explicit the requirement in classical factor models that the error terms have low intercorrelation. Note that for an arbitrarily given $t \in T$, e_t is orthogonal to e_s except for a λ-null set of $s \in T$ with $s \neq t$. This λ-null set may often contain many points. It may also happen that some unsystematic risk is uncorrelated with every asset, and thus has no presence at all in the financial market. The first part of Condition (iv) is a strong version of the law of large numbers for the process, and captures in an exact sense the common-sensical notion that unsystematic risks can be completely canceled through diversification.[21] Indeed, it is this law of large numbers, and the cancellation it embodies, that allows complete diversification of unsystematic risks: as long as p is square integrable over T, almost all sample functions of the process $p(t)e(t,\omega)$ have zero means. Together with the Karhunen-Loéve type biorthogonal expansion, it is crucial for the viability of our distinctive (exact arbitrage) approach to the APT.

For any portfolio p, Condition (iv) yields[22]

$$\begin{aligned} \mathcal{R}_p(\omega) &= \int_T p(t)\mu(t)d\lambda + \sum_{n=1}^{\infty} \lambda_n \left(\int_T p(t)\psi_n(t)d\lambda \right) \varphi_n(\omega) \\ &= (p,\mu) + \sum_{n=1}^{\infty} \lambda_n (p,\psi_n)\varphi_n(\omega). \end{aligned} \tag{3}$$

[20] The corresponding continuous analogue for processes which are continuous in quadratic means on an interval is often called the Karhunen-Loéve expansion theorem and is well-known; see [31]. Note that it is a trivial matter to require the factors to be orthonormal, but non-trivial to show that both factors and factor loadings can be orthogonal among themselves; see [41, 42] for details.

[21] In the terminology of [7, p.1306], all portfolios are "well-diversified" since they contain only factor variance and no idiosyncratic variance; also [7, Footnote 3].

[22] Recall that (p,μ) and (p,ψ_n) denote the inner products $\int_T p(t)\mu(t)d\lambda$ and $\int_T p(t)\psi_n(t)d\lambda$.

Hence, by the fact that $\varphi_n,\ n \geq 1$, are orthonormal with means zero,

$$E(p) = (p, \mu) = \int_T p(t)\mu(t)d\lambda; V(p) = \sum_{n=1}^{\infty} \lambda_n^2 (p, \psi_n)^2 = \sum_{n=1}^{\infty} \lambda_n^2 \left(\int_T p(t)\psi_n(t)d\lambda \right)^2. \quad (4)$$

In particular, p is a riskless portfolio, which is to say that $V(p) = 0$, if and only if p is orthogonal to all of the ψ_n.

Condition (v) and the second part of Condition (iv) say that in so far as integrals are concerned, one can simply ignore the error terms and focus on the factors. Conditions (iii) and (v) imply that for λ-almost all $t \in T$, e_t is orthogonal to all the $\varphi_n, n \geq 1$, as well as to e_s for λ-almost all $s \in T$. By ignoring a null set of assets, one can assume for convenience that this observation holds for *all* $t \in T$. With this assumption, it is obvious that for each $t \in T$, e_t is uncorrelated to x_s for λ-almost all $s \in T$ and hence is an unsystematic risk. For a centered random variable α defined on the sample space Ω, Condition (v) implies that for λ-almost all $t \in T$, α is uncorrelated with x_t if and only if α is orthogonal to all the factors φ_n. This means that a risk is unsystematic if and only if it is uncorrelated with all the factors. Thus, any non-trivial random variable α in the linear space $\mathcal{F}$ spanned by all the factors $\varphi_n, n \geq 1$, cannot be an unsystematic risk, and hence α must be correlated with a non-negligible segment of the financial market. In fact, for each $n \geq 1$,

$$\text{cov}\,(x_t, \varphi_n) = \int_\Omega f_t(\omega)\varphi_n(\omega)dP = \lambda_n \psi_n(t) \neq 0 \quad (5)$$

holds on a nonnull subset of T. It is then natural to define systematic risks to be the random variables which belong to the space $\mathcal{F}$, and thus the conditional expectation $E(f|_{\mathcal{U}})$ expresses the ensemble of the systematic risks for all assets. Formally

Definition 3 *A centered random variable β on the sample space Ω is said to be a systematic risk for a financial market x if β has finite variance and is in the linear space $\mathcal{F}$ spanned by all the factors $\varphi_n, n \geq 1$.*

In summary, a given risk γ can be additively decomposed into an element β in the endogenously identified space $\mathcal{F}$, and an element α in its orthogonal complement – Definitions 2 and 3 simply provide a crystallization of the common intuition that generally one can divide risks into a systematic and an unsystematic portion.[23]

We conclude this subsection by presenting an optimality property of the endogenously extracted factors. The result simply says that if one is allowed to use only m sources of risk to measure the systematic behavior of the market, the best approximation is achieved by

[23] According to the intuitive discussion in [37, pp. 293-294], as long as $\beta \neq 0$, one should call γ a systematic risk. On the other hand, our formal definition does not involve any unsystematic portion, i.e., it is purely systematic. For convenience, we shall observe the convention of referring to γ and β as a risk and a systematic risk respectively, rather than as a systematic risk and a purely systematic risk.

picking the m eigenfunctions corresponding to the largest m eigenvalues of the autocorrelation function of the asset return process. We can thereby justify the use of a relatively small number of factors, as defined in this hyperfinite framework, to understand the relevant correlation structures. There are of course, many ways of choosing factors to describe such structures, but the above procedure, based on the ranking of the importance of a particular factor in terms of the magnitude of its eigenvalue, is optimal in a well-specified sense.

Proposition 1 *For* $1 \leq i \leq m$, *let* $\mu_i \in \mathbb{R}$, $a_i \in \mathcal{L}^2(\lambda)$ *with* $\int_T a_i^2(t)d\lambda = 1$, $b_i \in \mathcal{L}^2(P)$ *with* $\int_\Omega b_i^2(\omega)dP = 1$. *Then*

$$\int\int_{T\times\Omega}[\sum_{i=1}^{m}\mu_i a_i(t)b_i(\omega) - f(t,\omega)]^2 d\lambda\otimes P \geq \sum_{n=m+1}^{\infty}\lambda_n^2 + \int\int_{T\times\Omega}[f - E(f|_U)]^2 d\lambda\otimes P,$$

or alternatively,

$$\Delta \equiv \int\int_{T\times\Omega}[\sum_{i=1}^{m}\mu_i a_i(t)b_i(\omega) - E(f|_U)]^2 d\lambda\otimes P \geq \sum_{n=m+1}^{\infty}\lambda_n^2.$$

The minimum is achieved at $\mu_i = \lambda_i$, $a_i = \psi_i$, $b_i = \varphi_i$ *for* $1 \leq i \leq m$. *If* λ_m *is an eigenvalue of unit multiplicity, and if the minimum is achieved by* $\sum_{i=1}^{m}\mu_i a_i(t)b_i(\omega) \equiv \beta(t,\omega)$, *then* $\beta(t,\omega) = \sum_{n=1}^{m}\lambda_n\varphi_n(\omega)\psi_n(t)$.

The result also indicates that even though one can use m sources of risk to approximate the ensemble of risks in the market in an optimal way, unsystematic risks remain irrespective of the size of m. On the other hand, since unsystematic risks have no significance from a macroscopic point of view, one can simply choose m so that the square sum of the remaining scaling constants is small enough.

2.2 Exact Arbitrage and APT

We begin with our rendition of Ross' theorem on asset pricing in large asset markets without arbitrage, but in contrast to his treatment, we work with an endogenous factor structure, the asset pricing formula is exact and, implies, as well as implied by, a no exact arbitrage assumption. We begin with a precise formulation of such an assumption; namely, the common-sensical assertion that a riskless and costless portfolio earns a zero rate of return.

Definition 4 *A market does not permit exact arbitrage opportunities if for any portfolio* p, $V(p) = C(p) = 0$ *implies* $E(p) = 0$.

We are now ready to state a theorem on the equivalence of the validity of an APT type pricing formula with the economic principle of no arbitrage. The formula simply says that except for a null set of assets, the expected return of an asset is linearly dependent on its

factor loadings in an exact way. To show that asymptotic no arbitrage implies an APT linear equation, Huberman [18] uses the projection of the expected return function onto the closed subspace generated by the factor loadings ψ_n together with the constant function 1, and to which a relevant orthogonal vector becomes arbitrary small as the number of assets becomes arbitrary large. Here we use a similar idea to show the necessity part of Theorem 1; however, the analogue of the relevant vector is identically zero in our setting.[24] Such a kind of exact equality leads to the discovery that the exact no arbitrage condition is also implied by the relevant APT linear equation. The details are as follows.

Theorem 1 *A market does not permit exact arbitrage opportunities if and only if there is a sequence* $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that for* λ*-almost all* $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$.

Proof: We begin with necessity. For an arbitrary portfolio p, let p_r be the projection of p on the closed subspace spanned by the constant function 1 and all the ψ_n. Denote $p_s = p - p_r$. Since $\mu \in \mathcal{L}^2(\lambda)$ from (2) above, we can also project it on the same closed subspace, and define μ_r and μ_s accordingly. If p is costless and riskless, then it is clear from Definition 1 and (4) above that p is orthogonal to 1 and to all of the ψ_n. This implies that $p_r = 0$. In this case, we obtain that

$$E(p) = \int_T p_s(t)\mu(t)d\lambda(t) = \int_T p_s(t)\mu_s(t)d\lambda(t).$$

Thus, no arbitrage means that $\int_T p_s(t)\mu_s(t)d\lambda(t) = 0$ for any p_s, and in particular, it is true when $p_s = \mu_s$. Hence, we obtain $\int_T \mu_s^2(t)d\lambda = 0$, and thus $\mu_s(t) = 0$ for λ-almost all $t \in T$. By the definition of μ_r, there are real numbers $\{\tau_n\}_{n=0}^{\infty}$ such that $\mu(t) = \mu_r(t) = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$ for λ-almost all $t \in T$.

On the other hand, for the sufficiency part of the claim, the validity of the arbitrage pricing formula clearly implies $\mu_s = 0$, which also furnishes us the no arbitrage condition. ∎

As defined in the above proof, μ_s is the orthogonal complement of μ on the subspace spanned by the constant function 1 together with all the factor loadings ψ_n. The following corollary is obvious.

Corollary 1 *A market does not permit exact arbitrage opportunities if and only if* $\mu_s = 0$.

If one is only allowed to use finitely many factors among countably many factors, then the following obvious corollary says that one can still obtain an approximate APT pricing result.

Corollary 2 *If a market does not permit exact arbitrage opportunities, there is a sequence* $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that* $\lim_{k\to\infty} \|\mu_t - \tau_0 - (\sum_{n=1}^{k} \tau_n \psi_n(t))\|_2 = 0$, *where* $\|\cdot\|$ *is the norm in the Hilbert space* $\mathcal{L}^2(\lambda)$.

[24] As such, the proof of the result bypasses any sequential arguments and simply formalizes Ross' [35, 36] heuristics; also see [18, 19] and [38, Theorem 2; p. 118], the latter in the context of what is termed an approximate factor structure.

2.3 The Equivalence of Beta Pricing and APT Pricing

In this section we connect the APT pricing formula to a beta model for asset pricing in which the expected return on any asset is linearly related to the rate of return on a reference portfolio, as exposited, for example, in Duffie's textbook [12, Chapter 1]. Note, in passing, that the beta model does not introduce investors' preferences explicitly as in the CAPM model of Sharpe [40] and Lintner [28]; and thus the prediction that the reference portfolio is held by optimizing agents is not a consequence of the beta model.[25]

Before turning to the equivalence between the APT pricing formula and one deriving from the beta model, we need to specify a particular portfolio h. This portfolio is defined by projecting the constant function 1 on the closed subspace orthogonal to that spanned by $[\psi_1(\cdot), \psi_2(\cdot), \cdots]$. For each $n \geq 1$, let $s_n = \int_T \psi_n(t) d\lambda(t) = (1, \psi_n)$, and note that h is given by $h(t) = 1 - \sum_{n=1}^{\infty} s_n \psi_n(t)$. Since h is orthogonal to the $\psi_n(\cdot)$, we obtain

$$(h,h) = \int_T h^2(t) d\lambda = (h,1) = \int_T h(t) d\lambda = C(h) \equiv h_0. \tag{6}$$

We can now present

Theorem 2 *The following two conditions are equivalent:*

(i) there is a portfolio M and a real number ρ such that for λ-almost all $t \in T$, $\mu_t = \rho + cov\,(x_t, M)$, where we follow the convention that the covariance $cov\,(x_t, M)$ is actually $cov\,(x_t, \mathcal{R}_M)$;

(ii) there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$, and for λ-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$.

Proof: For $(i) \Longrightarrow (ii)$, let $M(t) = M_0 h(t) + \sum_{n=1}^{\infty} M_n \psi_n(t) + M_s(t)$ such that M_s is orthogonal to h and all the ψ_n. Then by (3), $\mathcal{R}_M - E(M) = \sum_{n=1}^{\infty} \lambda_n M_n \varphi_n$. Since M is square integrable, Bessel's inequality [39, p. 88] guarantees that $\sum_{n=1}^{\infty} M_n^2 \leq (M,M) < \infty$. It is also clear that $\mathcal{R}_M - E(M)$ is square integrable, and thus we can then appeal to Condition (v) of Theorem A to assert that for λ-almost all $t \in T$,

$$\text{cov}\,(x_t, M) = \int_\Omega (x_t - \mu_t)(\omega)\,(\mathcal{R}_M(\omega) - E(M))\,dP = \sum_{n=1}^{\infty} \lambda_n^2 M_n \psi_n(t).$$

Hence it follows from (i) above that

$$\mu_t = \rho + \sum_{n=1}^{\infty} \left(\lambda_n^2 M_n\right) \psi_n(t).$$

[25] We are indebted to an Associate Editor and a referee for their emphasis on these observations.

By letting $\tau_0 = \rho$ and $\tau_n = \lambda_n^2 M_n$ for $n \geq 1$, we obtain the APT linear pricing equation. Since $\sum_{n=1}^{\infty} M_n^2 < \infty$, we have $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$.

Next, we consider $(ii) \Longrightarrow (i)$. Assume that the APT linear pricing equation in (ii) holds, and $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$. Define a portfolio M as follows:

$$M(t) = \sum_{n=1}^{\infty} \left(\frac{\tau_n}{\lambda_n^2}\right) \psi_n(t).$$

By $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$, we know that M is λ-square integrable, and hence is a well-defined portfolio in our setting. By (3), it is easy to see that the difference of $\mathcal{R}_M$ with its mean is $\sum_{n=1}^{\infty}(\lambda_n(\tau_n/\lambda_n^2))\varphi_n(\omega)$. Thus, by Condition (v) of Theorem A, we obtain that for λ-almost all $t \in T$,

$$\begin{aligned} \text{cov}\,(x_t, M) &= \int_{\Omega} (x_t - \mu_t)(\omega)(\mathcal{R}_M(\omega) - E(M))dP \\ &= \sum_{n=1}^{\infty} (\lambda_n \psi_n(t)) \left(\frac{\tau_n}{\lambda_n}\right) = \sum_{n=1}^{\infty} \tau_n \psi_n(t). \end{aligned}$$

Let $\rho = \tau_0$. Hence, by the APT linear pricing equation in (ii), we obtain $\mu_t = \rho + \text{cov}\,(x_t, M)$ for λ-almost all $t \in T$, which is to say that (i) holds. ∎

If only finitely many factors are derived from the process x of asset returns, then the summability condition[26] in Theorem 2 (ii) is trivially satisfied. Otherwise, it means that the coefficient τ_n related to the risk premium of the n-th factor must be comparatively much smaller than the associated scaling constant. If one can earn a relatively large amount of premium by holding small factor risks, then the risk premium awarding scheme in the market cannot be described by a pricing formula based on a reference portfolio as discussed here.

2.4 APT and Beta Pricing with Observable Parameters

Theorem 2 exhibits the equivalence between APT and beta pricing without a concrete specification of the reference carrier portfolio M. In this section, we develop some notation to translate the coefficients τ_n of Theorems 1 and 2 into directly observed market parameters. Towards this end, consider the portfolio h defined in the previous subsection, and use it to define the following parameter μ_0 by

$$\mu_0 = \begin{cases} \frac{\int_T \mu(t)h(t)d\lambda}{\int_T h^2(t)d\lambda} = \int_T \mu(t)h(t)d\lambda(t)/h_0 & \text{if } h \not\equiv 0, \\ 0 & \text{if } h \equiv 0. \end{cases} \tag{7}$$

Finally, for each $n \geq 1$, let $\mu_n = (\mu, \psi_n) = \int_T \mu(t)\psi_n(t)d\lambda$, and note

$$\mu = \mu_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n + \mu_s. \tag{8}$$

[26]The summability condition follows from the existence of the reference portfolio M by Bessel's inequality.

If we view ψ_n as a portfolio, μ_n and s_n are respectively the mean $E(\psi_n)$ and cost $C(\psi_n)$ of ψ_n. Note also that the spaces respectively spanned by $[h, \psi_1(\cdot), \psi_2(\cdot), \cdots]$ and $[1, \psi_1(\cdot), \psi_2(\cdot), \cdots]$ are the same. The following corollary relates the τ_n in Theorem 1 to the s_n and μ_n.

Corollary 3 *Assume that for λ-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$. Then we have the following:*

(i) if $h \not\equiv 0$, then $\tau_0 = \mu_0$ and $\tau_n = \mu_n - \mu_0 s_n$ for $n \geq 1$;

(ii) if $h \equiv 0$, then $\tau_n = \mu_n - \tau_0 s_n$, where τ_0 is an arbitrary real number. In particular, one can take $\tau_0 = \mu_0 = 0$ and $\tau_n = \mu_n$ for $n \geq 1$.

Proof: By the assumption, we have for λ-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$, and hence by substituting $1 = h + \sum_{n=1}^{\infty} s_n \psi_n$, we derive $\mu = \tau_0 h + \sum_{n=1}^{\infty} (\tau_n + \tau_0 s_n)\psi_n$. By Equation (8) and the fact that $\mu_s = 0$, we obtain that $\tau_0 h = \mu_0 h$, and $\tau_n + \tau_0 s_n = \mu_n$ for all $n \geq 1$. For (i), we simply note that $h \not\equiv 0$ implies that $\tau_0 = \mu_0$, and thus $\tau_n = \mu_n - \mu_0 s_n$ for all $n \geq 1$. For (ii), we only have to consider whether $\tau_n + \tau_0 s_n = \mu_n$ for all $n \geq 1$, since $\tau_0 h = \mu_0 h = 0$ is always satisfied in this case irrespective of τ_0. Therefore, one can take $\tau_n = \mu_n - \tau_0 s_n$ for all $n \geq 1$ with τ_0 being a variable. ∎

The next corollary then characterizes the validity of the beta linear pricing equation in terms of the market parameters s_n and μ_n.

Corollary 4 *Assume that the asset market does not permit exact arbitrage opportunities. Then there is a portfolio M and a real number ρ such that for λ-almost all $t \in T$, $\mu_t = \rho + cov\,(x_t, M)$ if and only if one of the following holds:*

(i) if $h \not\equiv 0$, then $\sum_{n=1}^{\infty} (\mu_n - \mu_0 s_n)^2 / \lambda_n^4 < \infty$;

(ii) if $h \equiv 0$, then there exists a real number γ such that $\sum_{n=1}^{\infty} (\mu_n - \gamma s_n)^2 / \lambda_n^4 < \infty$.

Proof: Since there is no arbitrage, Corollary 3 implies that $\mu = \mu_0 + \sum_{n=1}^{\infty} (\mu_n - \mu_0 s_n)\psi_n$. If $h \not\equiv 0$, then such a representation is unique as shown in Corollary 3(i), and hence we can appeal to the equivalence of (i) and (ii) in Theorem 2, to obtain the desired equivalence. Thus (i) is shown.

For (ii), assume $h \equiv 0$. By Theorem 2, there is a portfolio M and a real number ρ such that for λ-almost all $t \in T$, $\mu_t = \rho + \mathrm{cov}\,(x_t, M)$ if and only if there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty} (\tau_n^2 / \lambda_n^4) < \infty$, and for λ-almost all $t \in T$, $\mu(t) = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$. On the other hand, if for λ-almost all $t \in T$, $\mu(t) = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$, then Corollary 3(ii) implies that $\tau_n = \mu_n - \tau_0 s_n$. Therefore, the equivalence follows by taking τ_0 to be some number γ. ∎

2.5 Essential, Inessential and Unsystematic Risks: A Tri-variate Decomposition

We now turn to the notion of *essential* risks and to an associated asset pricing theorem. The importance of the theorem comes from its claim that the market only rewards a risk which is essential and which can never be eliminated through any sort of diversification. It asserts that two assets with the same essential risks earn the same risk premium even though they hold different systematic risks. In other words, even though systematic risks are rewarded as indicated by the APT model and its associated pricing formula, the relevant inessential risks among them do not earn a premium. Thus, the fact that APT allows *multiple* sources of industry-wide or market-wide factor risks does not mean that the reward scheme for risk taking necessarily involves a multiple number of risks; the market, in fact, only rewards the holding of one type of risk as described by a particular portfolio I_0, and the risk premium for other types of risk depend on the correlation of these risks with the essential risk X_0 embodied in the return to this portfolio. Stated more simply, this formula shows that the usual APT claim that the market rewards only systematic risks is not sharp enough; systematic risks can be reduced still further until a portfolio has only essential risk, and it is only this component of systematic risk that earns a premium.

Towards the execution of these ideas, we shall work with a fixed index portfolio I_0 explicitly constructed from the factor loadings in order to measure the risk most relevant to the risk premium of any asset. Assume that $\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^4 < \infty$, and let

$$I_0 = \sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n^2)\psi_n, \text{ with net random return } X_0 = \sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n)\varphi_n. \quad (9)$$

We can check that the cost, mean and variance of I_0 are respectively given by $C(I_0) = \sum_{n=1}^{\infty} s_n(\mu_n - \mu_0 s_n)/\lambda_n^2$, $E(I_0) = \sum_{n=1}^{\infty} \mu_n(\mu_n - \mu_0 s_n)/\lambda_n^2$ and $V(I_0) = \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^2$. We can now use I_0 to present

Definition 5 *A centered random variable defined on the sample space Ω is an essential risk for a financial market x if it is in the one dimensional linear space generated by the net random return X_0 of the index portfolio I_0, and is an inessential risk if it is in the orthogonal complement of X_0 in the space $\mathcal{F}$ of systematic risks.*

The above definition formalizes our contention that a systematic risk can be additively decomposed into an essential risk and an inessential risk.

We now turn to the pricing formula based on portfolio I_0 and one that only rewards the holding of essential risks. If $V(I_0) \neq 0$, we define the beta β_t of asset t by

$$\beta_t = \frac{\text{cov}\,(x_t, I_0)}{V(I_0)} = \frac{\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n(t)}{\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^2}.$$

For each asset t, let

$$Y(t,\omega) = \sum_{n=1}^{\infty} \left(\lambda_n \psi_n(t) - \frac{\beta_t}{\lambda_n}(\mu_n - \mu_0 s_n) \right) \varphi_n(\omega).$$

Then it is easy to check that Y_t is orthogonal to X_0 and the portion of systematic risk $\sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n$ in the total risk f_t of asset t can be written as the sum of an essential risk $\beta_t X_0$ and an inessential risk Y_t. Hence

$$x(t,\omega) - \mu(t) = \beta_t X_0(\omega) + Y(t,\omega) + e(t,\omega).$$

That is, we can break down the total risk $x_t - \mu_t$ of an asset t into three components: the first involves the projection of the risk onto X_0, the component of essential risk; the second is the projection on the orthogonal complement of X_0 in the space of systematic risks, the component of inessential risk; and the third is the component of residual unsystematic risk. The following theorem shows that as long as there is no arbitrage, the risk premium of almost all assets t only depends on β_t, the level of essential risk held in the asset. More precisely, it is equal to the beta of the asset multiplied by the risk premium of the index portfolio. Note that the equivalence of (i) and (ii) below is simply a restatement of Theorem 1, which is included here to emphasize the unification, in our setting, of APT, the beta pricing model and the no arbitrage condition.

Theorem 3 *Assume that* $\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^4 < \infty$ *and the market is non-trivial in the sense that the expected return function* μ *is not essentially the constant function* μ_0. *Then the following conditions are equivalent:*

(i) the market does not permit exact arbitrage opportunities;

(ii) there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that for* λ*-almost all* $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$;

(iii) for λ*-almost all* $t \in T$, $\mu_t = \mu_0 + \beta_t \left(E(I_0) - \mu_0 C(I_0) \right)$.

Proof: The equivalence of (i) and (ii) is already shown in Theorem 1. To show the rest, note that the relevant formulae for the cost, return and variance of the index portfolio I_0 imply that $V(I_0) = E(I_0) - \mu_0 C(I_0)$, and hence by the definition of β_t, we obtain

$$\mu_0 + \beta_t \left(E(I_0) - \mu_0 C(I_0) \right) = \mu_0 + \sum_{n=1}^{\infty} (\mu_n - \mu_0 s_n)\psi_n(t).$$

If (ii) holds, then Corollary 3 implies that for λ-almost all $t \in T$, $\mu(t) = \mu_0 + \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n(t)$, and hence $\mu(t) = \mu_0 + \beta_t \left(E(I_0) - \mu_0 C(I_0) \right)$ for λ-almost all $t \in T$, i.e., (iii) holds.

On the other hand, if we assume (iii), then the computation in the first paragraph implies that (ii) holds. Therefore all the three statements are equivalent. ∎

3 Conceptual Ramifications for APT

In this second substantive section of this paper, we provide an analysis of our results in Section 2 in the broader context of the motivations behind the APT literature. In particular, we relate our notion of *no exact arbitrage* to the more conventional asymptotic formulation, termed here *no asymptotic arbitrage.*[27] We also consider an asymptotic formulation of the no exact arbitrage assumption of this paper, termed here no *uniformly-asymptotic arbitrage.*

3.1 Asymptotic versus Exact Arbitrage

We begin by drawing the reader's attention to the fact that the assumptions of no exact arbitrage and no asymptotic arbitrage are both made on an asset market as discusses in Section 2 above. Thus, the net return of asset t at sample realization ω is $x_t(\omega) - \mu_t = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega) + e_t(\omega)$ as in Theorem A. Without loss of generality, assume that the return x_t of any asset $t \in T$ is in $\mathcal{L}^2(P)$. Thus $\sum_{n=1}^{\infty} \lambda_n^2 \psi_n^2(t) < \infty$ for every $t \in T$, and in particular, $\psi_n(t)$ is finite for every n and every t. An overview of the results of this subsection are sketched in Figure 1.

As in much of the previous APT literature referred to above (see [27]), the no asymptotic arbitrage assumption only involves finite portfolios. Such a portfolio p is a function $\alpha : T \longrightarrow \mathbb{R}$ for which $\alpha(t) \neq 0$ for only finitely many t. In this case, one can simply write $C(p) = \sum_{t\in T} \alpha_t$, $E(p) = \sum_{t\in T} \alpha_t \mu_t$, $R(p) = \sum_{t\in T} \alpha_t x_t$ and

$$V(p) = V\left(\sum_{n=1}^{\infty} \sum_{t\in T} \alpha_t \lambda_n \psi_n(t)\varphi_n(\omega) + \sum_{t\in t} \alpha_t e_t(\omega)\right).$$

When the unsystematic risk terms e_t are orthogonal to all the factors φ_n, it is obvious that $V(p) = \sum_{k=1}^{\infty} \left(\sum_{t\in T} \alpha_t \lambda_n \psi_n(t)\right)^2 + V(\sum_{t\in T} \alpha_t e_t(\omega))$. Since only finitely many terms are non-zero in all the sums $\sum_{t\in T}$ involving α_t, they are all well-defined. We present the formal definition of the no asymptotic arbitrage assumption below.

Definition 6 *A market does not permit asymptotic arbitrage opportunities if for any sequence of finite portfolios* $\{p_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} V(p_n) = 0$ *and* $\lim_{n\to\infty} C(p_n) = 0 \Longrightarrow \lim_{n\to\infty} E(p_n) = 0$.

We can now present

Proposition 2 *If a market with a riskless asset* x_{t_0}, $t_0 \in T$, *of a positive return* ρ *does not permit asymptotic arbitrage opportunities, it does not permit exact arbitrage opportunities.*

[27] See the references in Footnote 4 above.

Proof: By Proposition 1 in [7] and [26], the cost functional is continuous, and thus there is a random variable[28] C with a finite second moment such that the inner product $(C, x_t) = 1$ for all $t \in T$. In particular, $(C, x_{t_0}) = 1$, and thus $(C, 1) = E(C) = (1/\rho)$. Let μ_r and μ_s be the projections as in the proof of Theorem 1 above. Then Theorem A(v) and $(C, x_t) = 1$ for all $t \in T$ imply

$$1 = \mu_t(C, 1) + \sum_{n=1}^{\infty} \lambda_n (C, \varphi_n)\psi_n(t) + (C, e_t) = \mu_r/\rho + \mu_s/\rho + \sum_{n=1}^{\infty} \lambda_n (C, \varphi_n)\psi_n(t)$$

for λ-almost all $t \in T$, and hence μ_s is in the closed linear space spanned by $\{1, \psi_1, \cdots, \psi_n, \cdots\}$. On the other hand, $\mu_s \perp \{1, \psi_1, \cdots, \psi_n, \cdots\}$. Therefore, $\mu_s = 0$. Hence, by Corollary 1, the market does not permit exact arbitrage opportunities. ∎

The following example shows that the assumption of the absence of asymptotic arbitrage opportunities is strictly stronger than the assumption of the absence of exact arbitrage opportunities.

Example 1 For each $i = 1, 2, \ldots$, let e_i be a random variable with mean zero and variance $1/i$. Assume that the e_i's are mutually orthogonal. In an asset market consisting of risky assets $\{x_t\}_{t \in T}$, take a sequence $\{t_i\}_{i=1}^{\infty}$ from T such that

$$x_{t_i} = \mu_i + e_i, \quad \mu_i = \rho + \frac{1}{i}, \ i = 1, 2, \cdots,$$

with x_{t_0} a riskless asset with a positive return ρ, and $x_t = \rho$ for $t \neq t_i$, $i = 1, \cdots$. Since $E(x_t) = \rho$ for all except countably many assets, Theorem 1 implies that the market does not permit exact arbitrage opportunities. For each $n \geq 1$, take a portfolio $p_n = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ with $\alpha_0 = -\sum_{i=1}^{n} \alpha_i$ and $\alpha_i = i/n$ for $1 \leq i \leq n$, where α_i is the share of asset t_i. Then, it is obvious that $C(p_n) = 0$. It is also easy to obtain that

$$V(p_n) = \sum_{i=1}^{n} \alpha_i^2 V(e_i) = \sum_{i=1}^{n} 1/n^2 = 1/n$$

and

$$E(p_n) = \alpha_0 \rho + \sum_{i=1}^{n} \alpha_i \mu_i = \sum_{i=1}^{n} \alpha_i / i = 1.$$

Since $\lim_{n \to \infty} V(p_n) = 0$, and for all $n \geq 1$, $C(p_n) = 0$ and $E(p_n) = 1$, the market does permit asymptotic arbitrage. ∎

For our next result, we develop some additional terminology.

[28] As detailed in [26], the symbol C does triple duty: as the cost of a portfolio, as a continuous functional on the space of random returns and as a random variable in the space of random returns. Thus $E(C)$ is well-defined. We also remind the reader that (x, y) denotes the inner product between the random variables x and y.

Definition 7 *A market exhibits an APT asset-pricing formula with bounded total square deviations if there exists a sequence* $\{\tau_n\}_{n=0}^{\infty}$ *of real numbers such that* $\sum_{n=1}^{\infty}\tau_n\psi_n(t)$ *converges for every* $t \in T$ *and*

$$\sum_{t\in T}(\mu_t - \tau_0 - \sum_{n=1}^{\infty}\tau_n\psi_n(t))^2 < \infty.$$

We can now state, in the notation of this paper and for the sake of completeness, a consequence of the absence of asymptotic arbitrage opportunities in terms of an APT asset pricing formula with bounded total square deviations. The proof is omitted.[29]

Theorem 4 *Consider a market with a riskless asset of positive return* ρ *in which* $cov\,(e_s, e_t) = 0$ *for all* $s, t \in T$, *with* $s \neq t$ *and there exists* $0 \leq \zeta < \infty$ *such that* $V(e_t) \leq \zeta$ *for all* $t \in T$. *If such a market does not permit asymptotic arbitrage opportunities, then the market exhibits an APT asset-pricing formula with bounded total square deviations.*

Remark 1: Theorem 4 is essentially taken from [27], and is a version of Ross's result transcribed for a market with an *arbitrary* index set of assets. In [27, Theorem 2, Corollary 2], following [7] and [35, p. 355], we present additional, and more general, versions of the above result on the implications of the absence of asymptotic arbitrage opportunities in terms of asset pricing in a market. Our point is made most simply in terms of the version embodied in Theorem 4 presented here.

We now present an example to show that the assumption of a market not permitting asymptotic arbitrage opportunities, while sufficient, is not necessary for the validity of the APT asset-pricing formula with bounded total square deviations. In other words, for an asset market with a strict factor structure, the absence of realizing gains from asymptotic arbitrage is strictly stronger than claiming the validity of the usual APT type formula in the literature.

Example 2 Use the same market as in Example 1. Then it is obvious that the market has no factors and exhibits an APT asset-pricing formula with bounded total square deviations. However, it does permit asymptotic arbitrage opportunities as shown in Example 1. ∎

For the next series of observations, we need an additional definition.

[29] As in the proof of Theorem 1 in [27], the corresponding τ_n can be taken to be $-\rho\lambda_n(C, \varphi_n)$, where C is the cost random variable as in the proof Proposition 2. Thus,

$$\sum_{n=1}^{\infty}|\tau_n\psi_n(t)| \leq \rho\left(\sum_{n=1}^{\infty}(C,\varphi_n)^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\lambda_n^2\psi_n^2(t)\right)^{1/2} < \infty.$$

Hence $\sum_{n=1}^{\infty}\tau_n\psi_n(t)$ converges for every $t \in T$. The rest of the proof is exactly the same as that of Theorem 1 in [27].

Definition 8 *A set $U \subseteq T$ of assets in a market is said to be exactly factor-priced if there exists a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty} \tau_n \psi_n(t)$ converges for every $t \in T$ and*

$$\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t) \text{ for all } t \in U.$$

We can now present two straightforward but useful observations.

Remark 2: If a market exhibits an APT asset-pricing formula with bounded total square deviations, then there exists a set S of countable cardinality such that all assets in $T - S$ are exactly factor-priced.

Remark 3: If in a market, all assets in $T - S$, S of countable cardinality, are exactly factor-priced, then all assets except those in a set of zero λ-measure are exactly factor-priced.

Note that in Remarks 2 and 3, we successively move from a stronger condition to a weaker one. In terms of the terminology of Definition 8, Theorem 1 says that a market does not permit exact arbitrage opportunities if and only if all assets except those in a set of zero λ-measure are exactly factor-priced. The remarks then also show that a market that exhibits an APT pricing formula with bounded total square deviations, or in which all but countably many assets are exactly factor-priced, does not permit exact arbitrage opportunities. As shown in Examples 3 and 4 below, the converse to both statements is false.

The following example shows the existence of a market that neither permits exact arbitrage opportunities nor exhibits the APT asset-pricing formula with bounded total square deviations

Example 3: For each $i = 1, 2, \ldots$, let e_i be a random variables with mean zero and variance $1/i$. Assume that the e_i's are mutually orthogonal. In an asset market consisting of risky assets $\{x_t\}_{t \in T}$, take a sequence $\{t_i\}_{i=1}^{\infty}$ from T such that

$$x_{t_i} = \mu_i + e_i, \;\; \mu_i = 1 + \rho, \; i = 1, 2, \cdots,$$

with x_{t_0} a riskless asset with a positive return ρ, and $x_t = \rho$ for $t \neq t_i, \; i = 1, \cdots$. Since $E(x_t) = \rho$ for all except countably many assets, Theorem 1 implies that the market does not permit exact arbitrage opportunities. It is easy to check that this market does not exhibit the APT asset-pricing formula with bounded total square deviations. ∎

The next example shows that there is a market that neither permits exact arbitrage nor prices all but countably many assets exactly in terms of factor pricing.

Example 4: Consider an uncountable set $A \subseteq T$ with λ-measure zero. Let ρ be a positive number and $e_t, \; t \in A$ an orthonormal set of random variables with mean zero. In an asset market consisting of risky assets $\{x_t\}_{t \in T}$, such that

$$x_t = 1 + \rho + e_t, \; t \in A,$$

and $x_t = \rho$ for $t \notin A$. There are no factors in this market. Since $E(x_t) = \rho$ for all assets except those in A, Theorem 1 implies that the market does not permit exact arbitrage opportunities. It is also not true that all but countably many assets are exactly factor-priced. ∎

We now state as a remark an equivalence result that is by now well-known and goes back to [7]; see also [34] and, in the context of a market with an index set of arbitrary cardinality, [27, Proposition 1].

Remark 4: A market with a riskless asset does not permit asymptotic arbitrage opportunities if and only if there exists a continuous functional, termed a cost functional, on the space of asset returns.

The following is a triviality.

Remark 5: If a cost functional C is defined on the returns x_t of all assets $t \in T$ (i.e., $(C, x_t) = 1$ for all $t \in T$), then it is defined on the returns x_t of all assets $t \in T - S$, $\lambda(S) = 0$. Informally, if a cost functional prices all assets, then it prices almost all assets.

We now present conditions under which there exists a cost functional that prices almost all assets.

Proposition 3 *Assume that $h \not\equiv 0$ and the market does not permit exact arbitrage opportunities. Then there is a continuous functional C such that $C(x_t) = 1$ for λ-almost all $t \in T$ if and only if $\mu_0 \neq 0$ and $\sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n)^2 < \infty$.*

Proof: Note that $h \not\equiv 0$ implies that the set $\{h, \psi_1, \ldots, \psi_n, \ldots\}$ is linearly independent, and so is $\{1, \psi_1, \ldots, \psi_n, \ldots\}$. As in Corollary 3, $\mu_t = \mu_0 + \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n(t)$. As noted in Footnote 28, we shall still use C to denote the pricing random variable corresponding to continuous functional C. Let $C = C_0 + \sum_{n=1}^{\infty} C_n\varphi_n + C_s$, where C_s is orthogonal to $1, \varphi_1, \ldots, \varphi_n, \ldots$. Then Bessel's inequality implies that $\sum_{n=1}^{\infty} C_n^2 < \infty$. It is easy to see that for λ-almost all $t \in T$, $(C, x_t) = 1$ if and only if for λ-almost all $t \in T$,

$$C_0\mu_0 + \sum_{n=1}^{\infty}[C_0(\mu_n - \mu_0 s_n) + \lambda_n C_n]\psi_n(t) = 1,$$

which is equivalent to

$$C_0\mu_0 = 1, C_0(\mu_n - \mu_0 s_n) + \lambda_n C_n = 0 \text{ for } n \geq 1$$

by the linear independence of $\{1, \psi_1, \ldots, \psi_n, \ldots\}$. It is then equivalent to

$$\mu_0 \neq 0 \text{ and } \sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n)^2 < \infty,$$

and the proof is complete. ∎

As noted in Remark 4, the no asymptotic arbitrage assumption is closely related to the existence of a continuous cost functional that prices all assets in a market with a riskless asset. However, the following three examples show that the existence of a continuous cost functional that prices almost all assets could be very different from the assumption of no exact arbitrage in a market with a riskless asset. The first example shows that in the case when $h \equiv 0$, the existence of a continuous functional that prices almost all assets does not imply the absence of exact arbitrage opportunities.

Example 5: Take functions ψ_1 and ψ_2 on T such that $\psi_1 \equiv 1$, $E\psi_2 = 0$ and $E\psi_2^2 = 1$. Then $1 - (E\psi_1)\psi_1 \not\equiv 0$ and $h \not\equiv 0$. Choose μ_s on T such that ψ_1, ψ_2, μ_s are orthonormal. Let e be a process on the Loeb product space such that the e_t's are almost surely orthogonal with mean zero and variance one.[30] Define an asset market on the Loeb product space with the following returns. Take $t_0 \in T$ and define $x_{t_0} \equiv \rho$ for some positive number ρ. For $t \neq t_0$,

$$x_t(\omega) = \psi_1(t) + \psi_2(t) + \mu_s(t) + \psi_1(t)\varphi_1(\omega) + \psi_2(t)\varphi_2(\omega) + e_t(\omega)$$

Let $C = \varphi_1$. Then $C(x_t) = (C, x_t) = \psi_1(t) = 1$ for λ-almost all $t \in T$; but the market permits exact arbitrage by Corollary 1. ∎

The next example shows that there may not exist a continuous functional that prices almost all assets even though the market does not permit exact arbitrage opportunities and h is non-zero.

Example 6: Let ψ and φ have mean zero and variance one on T and Ω respectively, and e a process on the Loeb product space such that the e_t's are almost surely orthogonal with mean zero and variance one. Define an asset market on the Loeb product space with the following returns. Take $t_0 \in T$ and define $x_{t_0} \equiv \rho$ for some positive number ρ. For $t \neq t_0$, let

$$x_t(\omega) = \psi(t) + \psi(t)\varphi(\omega) + e_t(\omega).$$

By Theorem 1, this market does not permit exact arbitrage. It is clear that $h \equiv 1$ and $\mu_0 = 0$. Proposition 3 implies that there is no continuous functional that prices almost all assets, even though there is a riskless individual asset x_{t_0}. ∎

Finally, we present an example of a market with $\mu_0 \neq 0$, which does not permit exact arbitrage opportunities but in which there is no continuous cost functional pricing almost all assets.

Example 7: Let $\rho > 0$, $\{\psi_n : 1 \leq n < \infty\}$, $\{\varphi_n : 1 \leq n < \infty\}$ be collections of orthonormal and centered random variables, and e a process on the Loeb product space such that the e_t's

[30] The existence of such almost surely orthogonal processes on atomless Loeb product spaces follows from Theorem 6.2 in [42].

are almost surely orthogonal with mean zero and variance one. Define an asset market on the Loeb product space with the following returns. Take $t_0 \in T$ and define $x_{t_0} \equiv \rho$ for some positive number ρ. For $t \neq t_0$, let

$$x_t(\omega) = \mu_t + \sum_{n=1}^{\infty}(1/n)\psi_n(t)\varphi_n(\omega) + e(t,\omega) \text{ and } \mu_t = \rho + \sum_{n=1}^{\infty}(1/n)\psi_n(t),$$

By Theorem 1, this market does not permit exact arbitrage opportunities. Now, it is clear that for this market $h \equiv 1,\ \mu_0 = \rho > 0$, and for all $n \geq 1,\ \lambda_n = \mu_n = (1/n)$ and $s_n = 0$. Then

$$\sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n)^2 = \sum_{n=1}^{\infty} 1 = \infty.$$

We now appeal to Proposition 3 to assert that there is no continuous functional C such that $C(x_t) = 1$ for λ-almost all $t \in T$. ∎

3.2 Exact Arbitrage: An Asymptotic Version

As noted in [41], the asymptotic properties of stochastic processes on large finite probability spaces are in general equivalent to certain properties of processes on the Loeb product spaces.[31] This metatheorem notwithstanding, one does get insight into the idealized limit case by translating the results presented above into the asymptotic setting. The motivation behind such an exercise has by necessity to be illustrative – it would be tedious to translate each result, with each of its associated formulas, into an approximate discrete setting of the asset space or the sample space or both.

Towards this end, consider a sequence of markets $\mathcal{M}_n,\ n = 1, 2, \cdots$, where in each market $\mathcal{M}_n$, there are n assets indexed by the set $T_n = \{1, 2, \cdots, n\}$, and endowed with the uniform probability measure λ_n on T_n. Each asset t in the n-th market has a unit cost and a one-period random return x_{nt}, a real-valued random variable on a fixed common probability space $(\Omega, \mathcal{A}, P)$. For notational simplicity, we shall regard x_n as a process on the product space $(T_n \times \Omega, \mathcal{T}_n \otimes \mathcal{A}, \lambda_n \otimes P)$, where $\mathcal{T}_n$ is the power set on T_n. Note that the collection $\{x_n\}_{n=1}^{\infty}$ is also called a triangular array of random variables. We shall make the usual assumption of uniform integrability on the processes x_n^2 which we reproduce for the reader's convenience.

Definition 9 *A sequence of real-valued functions $\{g_n\}_{n=1}^{\infty}$ is said to be uniformly integrable*[32] *if*

$$\lim_{N\to\infty} \sup_{n\geq 1} \int_{|g_n|\geq N} |g_n| d\nu_n = 0.$$

[31] This is a technical assertion pertaining to the *nonstandard* extension whereby a result for the idealized nonstandard model can be translated into a standard asymptotic one for a large but finite setting.

[32] See, for example, Hidenbrand [17].

A portfolio p_n in the n-th market is simply a vector in $\mathbb{R}^n$, but we can also regard it as a function on T_n. Since our focus is on the asymptotic properties of a sequence of markets, the definitions of cost and random return of a portfolio are phrased in terms of arithmetic averages rather than sums – this is the usual practice, for example, in general equilibrium theory, [5], [17], [2]. One can interpret the weight $1/n$ as a unit specifying a "small" amount needed to purchase an asset t. We shall often use the notation of integration on T_n instead of summation in order to emphasize that the results presented here are asymptotic interpretations of those presented above for the idealized limit model. Thus, the cost $C(p_n)$ of a portfolio p_n is simply $\int_{T_n} p_n(t)d\lambda_n$, and its random return $\mathcal{R}_{p_n}(\omega)$ is $\int_{T_n} p_n(t)x_{nt}(\omega)d\lambda_n(t)$, and the expected return $E(p_n)$ and the variance $V(p_n)$ of the portfolio are respectively the mean and variance of the random return $\mathcal{R}_{p_n}(\omega)$. For a random variable α on Ω, we use $\|\alpha\|_2$ as before to denote the square root of its second moment, i.e., $\|\alpha\|_2 = \left(\int_\Omega \alpha^2 dP\right)^{1/2}$. We shall only work with those sequences $\{p_n\}_{n=1}^\infty$ of portfolios for which $\{p_n^2\}_{n=1}^\infty$ is uniformly integrable.

We can now present an asymptotic version of the absence of exact arbitrage – its difference from the absence of asymptotic arbitrage opportunities in an idealized market (as presented in Definition 6 above) should be noted.

Definition 10 *We say that the sequence of markets does not permit uniformly-asymptotic arbitrage opportunities if for any sequence of uniformly square integrable portfolios,* $\{p_n\}_{n=1}^\infty$, $\lim_{n\to\infty} V(p_n) = 0$ *and* $\lim_{n\to\infty} C(p_n) = 0 \Longrightarrow \lim_{n\to\infty} E(p_n) = 0$.

Next, we develop the notation to illustrate how one can proceed to provide an asymptotic interpretation of Theorem 1. For a market with n assets, $\mathcal{M}_n$, let the t-th asset have an expected return $\mu_n(t) = \int_\Omega x_{nt}(\omega)dP$, and assume that

$$x_n(t,\omega) = \mu_n(t) + \sum_{i=1}^{K} \lambda_{ni}\psi_{ni}(t)\varphi_{ni}(\omega) + e_n(t,\omega), \tag{10}$$

where K is a natural number, the factors $\{\varphi_{ni}\}_{n=1}^\infty$ and the factor loadings $\{\psi_{ni}\}_{n=1}^\infty$ are uniformly square integrable for each i, and for each n, $\{\psi_{ni} : 1 \le i \le K\}$ and $\{\varphi_{ni} : 1 \le i \le K\}$ are orthonormal. In addition, assume that the residual terms $\{e_n\}_{n=1}^\infty$ are uniformly square integrable and approximately orthogonal in the sense that

$$\lim_{n\to\infty} \int_{t_1\in T_n} \int_{t_2\in T_n} \left[\int_{\omega\in\Omega} e_{nt_1}(\omega)e_{nt_2}(\omega)dP\right]^2 d\lambda_n d\lambda_n = 0.$$

We can now present

Proposition 4 *Assume that the sequence of markets does not permit uniformly-asymptotic arbitrage opportunities. Then there exist sequences of real numbers* $\{\tau_{ni}\}_{n=1}^\infty$ *for* $i = 0, 1, \ldots, K$, *such that*

$$\lim_{n\to\infty} \|\mu_n(t) - (\tau_{n0} + \sum_{i=1}^{K} \tau_{ni}\psi_{ni}(t))\|_2 = 0.$$

The methodology underlying the proof of the above proposition is by now standard in the literature on nonstandard analysis; see, for example, [2] and [42, Section 9]. We leave it to the reader to compare this asymptotic version with its idealized counterpart embodied in Theorem 1. We also refer her to [24, 25] for additional asymptotic interpretations of the results for the idealized hyperfinite market.

3.3 Well-Diversified Portfolios in Countable Asset Markets

Ross [36, pp. 195-197], in his original presentation of the APT, writes

> We will develop such a theory without the additional baggage of mean-variance theory and will ... refer to it as the arbitrage theory. Throughout we will assume that the number of assets, n, is sufficiently large to permit our arguments to hold. We will also assume that the noise vector is sufficiently independent to permit the law of large numbers to work. [In particular,] we have assumed that the arbitrage portfolio is sufficiently well diversified to permit us to use the law of large numbers to eliminate the noise term, ... and in effect, to eliminate the independent risk from the portfolio return.

The basic intuition then is to consider the limit of an equal weight n-asset portfolio and let the number n go to infinity so as to invoke the classical law of large numbers. However, in a nutshell, the difficulty with these heuristics is that Ross equates the error to zero and then appeals to a purely linear-algebraic argument. As such, he presumably has an idealized limit model in mind.[33] The problem is that in any well-diversified portfolio, we want to know the relative amount being invested in each asset and it is far from clear how these amounts can be discerned in a context when the number of assets n goes to infinity!

In [6, Section 3, Definition 1], Chamberlain focuses on the space of random returns and refers to the limit of the returns from a sequence of finite portfolios as the return of a well-diversified portfolio. However, unlike the treatment presented above, what is identified in [6] is not the portfolio itself but its random returns. It is precisely these considerations that are resolved by Definition 1 and the discussion following it.

We conclude this subsection by observing that a similar comment can be made regarding the results reported in Werner [45]. Following Duffie's [11, Exercise 9.8] suggestion to model a "well-diversified" portfolio in a market with a countable number of assets as a purely finitely additive measure on the set $\mathbb{N}$ of natural numbers, Werner uses Pettis integration and, as in [11], relies on the restatement in [13] of the classical law of large numbers (in terms of integrals with respect to a purely finitely additive measure on the set $\mathbb{N}$ of natural numbers) to show that "every perfectly diversified portfolio has no idiosyncratic risk".[34] The question again reduces to the operational meaning of the limiting portfolio. It is well-known that a purely finitely

[33] See [36, Footnote 11] in this connection. Also the quote from Chamberlain-Rothschild cited in the second paragraph of the introduction to this paper.

[34] See [45, Theorem 2] for a precise formulation. We refer the reader to [44] for difficulties pertaining to this restatement.

additive measure on $\mathbb{N}$ takes zero values on each integer, and since $\mathbb{N}$ represents the list of asset names, such a measure finesses the difficulty of how to measure the relative amounts of each asset in a portfolio. Here, of course, we make a point well-known and well-understood both in general equilibrium and growth theory.[35]

3.4 APT in the Lebesgue Setting

In [27], the authors present a sustained, and in many ways, a decisive critique of an attempt to construct an idealized limit model for the APT based on the unit Lebesgue interval as the set of asset names. This critique has three aspects. The first concerns an exact pricing formula for all assets except those in a set of zero Lebesgue measure. As delineated in [27, Section 3.2.3], in a setting with finite portfolios and in the absence of asymptotic arbitrage opportunities, such a "result" is a simple and sole consequence of the *linearity* of the cost functional, and can be seen, in a well-specified and precise sense, as strengthening the assumptions and weakening the conclusions of standard results in the APT literature; see Footnote 4 above. Indeed, based on a straightforward generalization of the methods of Ross, Chamberlain-Rothschild and others, one obtains a pricing result with bounded total square deviations for an arbitrary set of assets, without any appeal to a measure-theoretic structure. As noted in Remarks 2 and 3, this asset pricing statement can be weakened to some exact factor pricing statements.

The other two aspects of the critique go beyond a "so-called" exact APT pricing formula, and concern attempts at a theory with a deeper reach, one with portfolios constituted by a non-negligible set of assets.[36] Here, the failure of the framework is total. This strong claim is based on two overwhelming considerations [27, Section 3.2.1 and 3.2.2]: first, no meaning can be given to the notions of the mean and variance of a well-diversified portfolio; and second, the specification of a factor structure is not robust with respect to a permutation of the space of asset names. Specifically, in the context of the Lebesgue unit interval as the space of asset names, the authors furnish examples of a market (i) with a single factor in which the aggregate of idiosyncratic risks can take arbitrarily given values as its mean and variance, (ii) with three measure-theoretic factors, which under "renaming", emerges with either one or two measure-theoretic factors. Thus, portfolio analysis is simply rendered incoherent.

The fundamental reason for such an incoherence is by now well-understood – it goes back to Doob's 1937 warning that "the sample functions are too irregular to be useful",[37] when non-trivial unsystematic risks are introduced for most assets. In fact, there is no exact counterpart to the classical law of large numbers for a Lebesgue continuum of random variables,

[35] See [5] for the former, and [22] for the latter. The fact that the purely finitely additive measure-theoretic approach suffers from a "lack of limiting properties" is also well-known (see [1] and [27, Section 3.1]). Also, totally absurd results may be obtained in the purely finitely additive setting (see [44]).

[36] Attention is limited to finite portfolios in [1], and hence this attempt is not made there. The failure of the Lebesgue setting due to basic measurability considerations is "bypassed".

[37] See [8, Theorem 2.2], [9, p. 67]; also [21], [13], [44].

and therefore no way for the elimination of unsystematic risks. Indeed, the set of samples whose corresponding sample functions are Lebesgue measurable is *proven* to have probability zero in [44]. The reader has already seen in Section 2 that these pathologies totally disappear in the Loeb measure-theoretic setting.[38]

We conclude this section by briefly considering the commentary on the model presented in this paper in Al-Najjar [1, pp. 242-243]. The author writes

> Khan and Sun ... arrive at asset pricing and factor structure results which mirror the substance and economic interpretation of the results first reported in this paper.[39]

As can be seen even on a first reading, the results reported in Section 2 are based on the notion of the absence of exact (Definition 4), and in the asymptotic context, of uniformly-asymptotic arbitrage opportunities (Definition 10). Since these concepts are new to the literature, and certainly to [1], it is difficult to give a coherent meaning to his statement. Al-Najjar also singles out the following four original contributions of his paper: (i) exact factor pricing for almost all assets, (ii) optimal extraction of sets of factors based upon a "criterion of explanatory power", (iii) the decomposition of risk into factor risk and idiosyncratic risk, (iv) the use of an infinite-dimensional analogue of the variance-covariance matrix to derive such a decomposition.[40]

We have already referred to (i) above, and (iii) is nothing but simply the starting assumption of all factor models that the return generating process of assets can be written as a summation of two components – the relevant point is that the term labeled idiosyncratic risk is not idiosyncratic in the sense that it can be aggregated away in the Lebesgue setting. The claims under (ii) and (iv) relate to the extraction and optimality of the factors. It is worth noting that the result [41, Theorem 3] on the extraction of factors based on the eigenvalues of an infinite-dimensional analogue of the variance-covariance matrix (i.e., the associated autocorrelation function R in Section 2.1 of this paper) had already been published two years before [1].[41] Such an extraction in the finite case is nothing but well-known procedure of the classical principal components model,[42] and in the infinite case, equally well-known continuous analogue, called the Karhunen-Loéve biorthogonal expansion,[43] generalized to the hyperfinite case in [41] and [42]. Note that the procedure for the computation of eigenvalues and eigenfunctions even

[38] It is perhaps also worth noting that renaming of assets poses no difficulty as in the finite case since hyperfinite sets have all the formal properties of finite sets (see, for example, [20]). In particular, every meaningful bijection on T is measure-preserving.

[39] However, Al-Najjar did not refer to two earlier papers [41] and [25] that are relevant to [1] and to this paper.

[40] For all these points, see [1, pp. 242-243].

[41] The earlier paper [41] was ignored in [1], while a restatement of the results in [41] as in part of Theorem A here is claimed in [1] to "mirror" the results of [1].

[42] See, for example, the text book [15, Section 8.3] and [7]

[43] Indeed, this Karhunen-Loéve expansion is itself the continuous analogue of the classical principal components model for a finite population. The theorem has many applications in statistical factor analysis, pattern recognition, and other fields. For the theoretical development, see the textbooks [31, Chapter XI], [3, Appendix], [4, Section 7.8]; and for applications, see, for example, [4], [14], [32] and their references.

in the infinite dimensional case is standard.[44] The so-called "criterion of explanatory power" in [1] is precisely a version of this standard procedure of finding eigenfunctions successively. Note that for a general asset return generating process in which a finite factor structure may not exist, Proposition 1 of this paper shows that the first m factors in Theorem A do provide an optimal approximation procedure for any m. That is, if m factors are used for approximation, then the optimal one is to use $\varphi_1, \ldots, \varphi_m$. Such a kind of optimality result is not shown in [1] even for the simple case in which the market has only finitely many factors.[45]

Al-Najjar also cursorily comments on the results reported in Section 2 above. He writes

> Khan and Sun note that asset prices are determined by their exposures to a benchmark portfolio ... the existence of such a portfolio is a consequence of the continuity of the pricing function.

This comment is presumably a consequence of the claim in [1, p. 240, last paragraph] that continuity of the cost functional implies that it is constituted solely by factor risk. In the example in [27, p. 246], the falsity of this claim is shown through a market with a continuum of assets and a strict one-factor structure in which the associated price functional is constituted only by idiosyncratic risks. A clarification on this score is also available in [7]. Here we emphasize again that even though the usual assumption of no asymptotic arbitrage is closely related to the existence of a continuous cost functional that prices all assets, our assumption of no exact arbitrage is very different from the existence of a continuous cost functional that prices (even) almost all assets (see Examples 5-7 above).

4 Concluding Summary

The model of asset pricing presented in this paper furnishes a precise and fruitful formulation of the notion of a "well-diversified" portfolio under which the relative amounts invested in each of a "large" number of assets can be specified. The heuristics of such a notion are pervasive in the finance literature, but it has so far resisted a rigorous analytical treatment. The conceptual vocabulary pertaining to the cost, expected return and the variance of a "well-diversified" portfolio then allows a rigorous formulation of an exact no-arbitrage assumption that is both intuitively natural and analytically simple. This assumption is both necessary and sufficient for an APT pricing formula to hold.

[44] See, for example, the textbook [16, p. 281]. All these relevant results are ignored in [1].

[45] Note that Proposition 5 of [1] states that an optimal finite "sub-factor structure" exists in a market which is already assumed to have finitely-many factors. It does not show that the first L factors, $\tilde{\delta}_1^*, \ldots, \tilde{\delta}_L^*$ as extracted in [1], are the ones that provide the optimal approximate L-factor structure. Note that the set $\{\tilde{\delta}_1, \ldots, \tilde{\delta}_L\}$ of L factors in the proof of Proposition 5 of [1] is not the set of first L extracted factors $\{\tilde{\delta}_1^*, \ldots, \tilde{\delta}_L^*\}$. In addition, one may refer to Footnote 12 in [1] where it is explicitly stated that the author has not "been able to prove ... a stronger version ... in which the assumption [of] a K-strict factor structure is eliminated." Thus, even the existence of an optimal finite "sub-factor structure" is not shown in [1] for the interesting case in which the existence of a finite factor structure in the continuum asset market is not already assumed.

In the idealized limit model, an asset's rate of return can be decomposed into systematic (factor) and idiosyncratic (non-factor) risk based on factors that are endogenously extracted from the process of asset returns to obtain an analogue of the well-known Karhunen-Loéve biorthogonal expansion of continuous time stochastic processes. The exact law of large numbers allows the substantiation of the claim that "well-diversified" portfolios exhibit no unsystematic (non-factor) risk. The systematic (factor) risk can be further decomposed into *essential* and *non-essential* risk through a "properly defined" set of factors, and a premium is paid to asset risk only through the particular role of the factors in the definition of this *essential* risk. Thus, three types of risks and their roles in a large financial market are exactly distinguished. This explicit tri-variate decomposition allows us to sight previous work as being able to handle only two of these three types of risks, one pair at one time and in different settings. Finally, the choice of factors as stated in Theorem A is also shown to satisfy an optimality property that the first m factors always provide the best approximation.

These ideas are elaborated in the first part of the paper on "exact arbitrage, risk analysis and asset pricing". In the second part on "conceptual ramifications for the arbitrage pricing theory", we do several things. First, as illustrated in Figure 1, we show that the new concept of exact no-arbitrage is strictly weaker than the asymptotic no arbitrage assumption conventional in the APT literature. This has novel implications for the continuity of the cost functional as well as for various versions of the APT asset pricing formula. Second, we illustrate how the concepts and results for the idealized measure-theoretic setting translate to markets with a large but finite number of assets. In this context, we identify the notion of no-uniformly-asymptotic arbitrage as the relevant translation of the idealized no-exact arbitrage assumption. Third, we recapitulate the inadequacies of the attempt to obtain an analytically viable notion of a "well-diversified" portfolio that is based on finitely additive measures on the set of natural numbers. Finally, we observe how the authors' critique of the Lebesgue setting, as developed in [27], is totally blunted in the idealized context presented in this paper, both with regard to the very definition of a "well-diversified" portfolio as well to the robustness of the model with respect to the permutation of asset names.

5 Appendix

The proof of Proposition 1 hinges on the projections of the functions $a_i(\cdot)$ and $b_i(\cdot)$ on suitable subspaces of the corresponding Hilbert spaces.

Proof of Proposition 1: We begin the proof by deriving the second expression from the first. Towards this end, note from Loéve (1977 b; p. 16) that $E(g|\mathcal{U}) = 0$ for a fixed $g \in \mathcal{L}^2(\lambda \otimes P)$ implies $E(gh|\mathcal{U}) = 0$ for any $h \in \mathcal{L}^2(\lambda \otimes P)$, and therefore that $\int\int_{T\times\Omega} g.h d\lambda \otimes P = 0$. Hence

$$\int\int_{T\times\Omega} (g+h)^2 d\lambda \otimes P = \int\int_{T\times\Omega} g^2 d\lambda \otimes P + \int\int_{T\times\Omega} h^2 d\lambda \otimes P.$$

From the equality $E[(f - f|\mathcal{U})|\mathcal{U}] = 0$, we obtain

$$\int\int_{T\times\Omega} [\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) - f(t,\omega)]^2 d\lambda \otimes P =$$
$$\int\int_{T\times\Omega} [\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) - E(f|\mathcal{U})]^2 d\lambda \otimes P + \int\int_{T\times\Omega} [f - E(f|\mathcal{U})]^2 d\lambda \otimes P.$$

Next, we show in three steps that $\Delta \geq \sum_{n=m+1}^{\infty} \lambda_n^2$.

Step 1: For each $1 \leq i \leq m$, let

$$b_i(\omega) = b_i'(\omega) + b_i''(\omega),$$

where b_i' is the projection of b_i on the space spanned by the φ_n. Thus, for each i, b_i'' is orthogonal to all the φ_n. Hence

$$\begin{aligned}
\Delta &= \int\int_{T\times\Omega} \left[\sum_{i=1}^{m} \mu_i a_i(t) b_i'(\omega) - \sum_{n=1}^{\infty} \lambda_n \varphi_n(\omega)\psi_n(t) + \sum_{i=1}^{m} \mu_i a_i(t) b_i''(\omega)\right]^2 d\lambda \otimes P \\
&= \int_T \left(\int_\Omega \left[\sum_{i=1}^{m} \mu_i a_i(t) b_i'(\omega) - \sum_{n=1}^{\infty} \lambda_n \varphi_n(\omega)\psi_n(t)\right]^2 dP \right. \\
&\qquad \left. + \int_\Omega \left[\sum_{i=1}^{m} \mu_i a_i(t) b_i''(\omega)\right]^2 dP \right) d\lambda \\
&\geq \int_T \int_\Omega \left[\sum_{i=1}^{m} \mu_i a_i(t) b_i'(\omega) - E(f|\mathcal{U})\right]^2 dP d\lambda.
\end{aligned}$$

Step 2: Take an orthonormal basis $\{d_1, \ldots, d_q\}$ for the space $\mathrm{sp}[b_1', \cdots, b_m']$. Note that the b_i' are not assumed to be linearly independent and hence $q \leq m$. Hence we can write

$$\sum_{i=1}^{m} \mu_i a_i(t) b_i'(\omega) = \sum_{i=1}^{q} \nu_i c_i(t) d_i(\omega),$$

with $\{d_1, \ldots, d_q\}$ orthonormal, $\int_T c_i^2(t) d\lambda = 1$, $\nu_i \in \mathbb{R}$ for $1 \leq i \leq q$. Since the d_i are in the space spanned by the φ_n, we have

$$d_i = \sum_{n=1}^{\infty} (d_i, \varphi_j)\varphi_j(\omega) \text{ where } (d_i, \varphi_j) = \int_\Omega d_i(\omega)\varphi_j(\omega) d\lambda.$$

Hence we obtain

$$\begin{aligned}
\Delta &\geq \int_T \int_\Omega \left[\sum_{i=1}^{q} \nu_i c_i(t) \sum_{j=1}^{\infty} (d_i, \varphi_j)\varphi_j(\omega) - \sum_{n=1}^{\infty} \lambda_j \varphi_j(\omega)\psi_j(t) \right]^2 dP d\lambda \\
&= \int_T \int_\Omega \left[\sum_{j=1}^{\infty} \left(\sum_{i=1}^{q} \nu_i (d_i, \varphi_j) c_i(t) - \lambda_j \psi_j(t) \right) \varphi_j(\omega) \right] dP d\lambda \\
&= \int_T \int_\Omega \sum_{j=1}^{\infty} \left(\sum_{i=1}^{q} \nu_i (d_i, \varphi_j) c_i(t) - \lambda_j \psi_j(t) \right)^2 d\lambda \\
&= \sum_{j=1}^{\infty} \lambda_j^2 \int_T \left(\psi_j(t) - \sum_{i=1}^{q} \frac{\nu_i (d_i, \varphi_j)}{\lambda_j} c_i(t) \right)^2 d\lambda.
\end{aligned}$$

Step 3: Take an orthonormal basis $v_1, \cdots, v_p$ for the space $\mathrm{sp}[c_1, \cdots, c_q]$. Then $p \leq q$, and we can write

$$\sum_{i=1}^{q} \frac{\nu_i (d_i, \varphi_j)}{\lambda_j} c_i(t) = \sum_{i=1}^{p} \alpha_i^j v_i(t).$$

Now, note that

$$\begin{aligned}
\int_T \left(\psi_j(t) - \sum_{i=1}^{q} \alpha_i^j v_i(t) \right)^2 d\lambda &\geq \int_T \left(\psi_j(t) - \sum_{i=1}^{p} (\psi_j, v_i) v_i(t) \right)^2 d\lambda \\
&= \int_T \psi_j^2(t) - \sum_{i=1}^{p} (\psi_j, v_i)^2 \int_T v_i(t)^2 d\lambda \\
&= 1 - \sum_{i=1}^{p} (\psi_j, v_i)^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\Delta &\geq \sum_{j=1}^{\infty} \lambda_j^2 \left[1 - \sum_{i=1}^{p} (\psi_j, v_i)^2 \right] \\
&= \sum_{j=1}^{p} \lambda_j^2 \left[1 - \sum_{i=1}^{p} (\psi_j, v_i)^2 \right] + \sum_{j=p+1}^{\infty} \lambda_j^2 \left[1 - \sum_{i=1}^{p} (\psi_j, v_i)^2 \right]. \qquad (11)
\end{aligned}$$

Since $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq \lambda_{n+1} \geq \cdots$, we obtain

$$\begin{aligned}
\Delta &\geq \sum_{j=1}^{p} \lambda_p^2 \left[1 - \sum_{i=1}^{p} (\psi_j, v_i)^2 \right] + \sum_{j=p+1}^{\infty} \lambda_j^2 \left[1 - \sum_{j=1}^{p} (\psi_j, v_i)^2 \right] \qquad (12) \\
&= \sum_{j=p+1}^{\infty} \lambda_j^2 + \lambda_p^2 \sum_{i=1}^{p} \left[1 - \sum_{j=1}^{p} (\psi_j, v_i)^2 \right] - \sum_{j=p+1}^{\infty} \lambda_j^2 \sum_{i=1}^{p} (\psi_j, v_i)^2 \\
&\geq \sum_{j=p+1}^{\infty} \lambda_j^2 + \lambda_p^2 \sum_{i=1}^{p} \left[1 - \sum_{j=1}^{p} (\psi_j, v_i)^2 \right] - \sum_{j=p+1}^{\infty} \lambda_p^2 \sum_{i=1}^{p} (\psi_j, v_i)^2 \qquad (13)
\end{aligned}$$

$$= \sum_{j=p+1}^{\infty} \lambda_j^2 + \lambda_p^2 \sum_{i=1}^{p} \left[1 - \sum_{j=1}^{\infty} (\psi_j, v_i)^2 \right].$$

Note that $\sum_{j=1}^{\infty} < v_i, \psi_j > \psi_j$ is the projection of v_i on the space spanned by the ψ_n. Hence, $1 = \int_T v_i^2(t) d\lambda \geq \sum_{j=1}^{\infty} (\psi_j, v_i)^2$. This implies that $1 - \sum_{j=1}^{\infty} (\psi_j, v_i)^2 \geq 0$ for $1 \leq i \leq p$. Since $p \leq m$, we obtain $\Delta \geq \sum_{j=p+1}^{\infty} \lambda_j^2 \geq \sum_{j=m+1}^{\infty} \lambda_j^2$. The proof of the assertion is complete, and we turn to the uniqueness claim.

Note the fact that the minimum is achieved implies that all of the inequalities involved must become equalities. In particular

$$\Delta = \sum_{j=m+1}^{\infty} \lambda_p^2 \text{ if and only if } p = m \text{ and hence } q = m.$$

Since $\lambda_{m-1} > \lambda_m > \lambda_{m+1}$, we obtain from (12) that

$$\sum_{i=1}^{m} (\psi_j, v_i)^2 \ = \ 0 \text{ for } j \geq m+1. \tag{14}$$

Note that $p = m$, and $\lambda_j < \lambda_m$, for $j \geq m+1$. Furthermore, we obtain

$$\int_T v_i^2(t) d\lambda = \sum_{j=1}^{\infty} < \psi_j, v_i >^2 \ \text{ for } 1 \leq i \leq m. \tag{15}$$

(14) and (15) imply that each v_i, $1 \leq i \leq m$, is in the space $\mathrm{sp}[\psi_1, \cdots, \psi_m]$. From (12) and (13), we obtain

$$1 = \int_T \psi_j^2(t) d\lambda = \sum_{i=1}^{m} (\psi_j, v_i)^2 \text{ for } 1 \leq j \leq m-1 \tag{16}$$

since for such a j, $\lambda_j > \lambda_m$. By (15) and (16), $1 = \sum_{j=1}^{m} (\psi_j, v_i)^2$ for $1 \leq i \leq m$, which implies that

$$\begin{aligned} m &= \sum_{i=1}^{m} \sum_{j=1}^{\infty} (\psi_j, v_i)^2 \\ &= \sum_{i=1}^{m} (\psi_m, v_i)^2 + \sum_{j=1}^{m-1} \sum_{i=1}^{m} (\psi_j, v_i)^2 \\ &= \sum_{i=1}^{m} (\psi_m, v_i)^2 + \sum_{j=1}^{m-1} 1 \qquad \text{by (11)} \\ &= \sum_{i=1}^{m} (\psi_m, v_i)^2 + m - 1. \end{aligned}$$

This implies that

$$1 = \sum_{i=1}^{m} < \psi_m, v_i >^2 = \int_T \psi_m^2(t) d\lambda. \tag{17}$$

(16) and (17) imply that each of $\psi_1, \cdots, \psi_m$ is in the space $\mathrm{sp}[v_1, \cdots, v_m]$. Hence $\psi_1, \cdots, \psi_m$, span the same space as $v_1, \cdots, v_m$, and hence $c_1, \cdots, c_m$, and hence $a_1, \cdots, a_m$. Note that the c_i are in the

space spanned by $a_1, \cdots, a_m$. Hence there is a nonsingular $m \times m$ matrix A such that $(a_1, \cdots, a_m) = (\psi_1, \cdots, \psi_m)A$. The function

$$\beta(t,\omega) = (a_1, \cdots, a_m) \begin{pmatrix} \mu_1 b_1 \\ \vdots \\ \mu_m b_m \end{pmatrix} = (\psi_1, \cdots, \psi_m) A \begin{pmatrix} \mu_1 b_1 \\ \vdots \\ \mu_m b_m \end{pmatrix}.$$

Now let

$$A \begin{pmatrix} \mu_1 b_1 \\ \vdots \\ \mu_m b_m \end{pmatrix} = \begin{pmatrix} \tau_1 \eta_1(\omega) \\ \vdots \\ \tau_m \eta_m(\omega) \end{pmatrix},$$

where $\int_\Omega \eta_i^2(\omega) dP = 1$. Then

$$\beta(t,\omega) = \sum_{i=1}^{m} \tau_i \psi_i(t) \eta_i(\omega).$$

Since the following integral is minimized,

$$\begin{aligned}
&\int\int_{T\times\Omega} \left(\sum_{n=1}^{\infty} \lambda_n \varphi_n(\omega)\psi_n(t) - \sum_{n=1}^{m} \tau_n \eta_n(\omega)\psi_n(t) \right)^2 d\lambda \otimes P = \sum_{n=m+1}^{\infty} \lambda_n^2 \\
&= \int_T \int_\Omega \left(\sum_{n=1}^{m} (\lambda_n \varphi_n(\omega) - \tau_n \eta_n(\omega))\psi_n(t) + \sum_{n=m+1}^{\infty} \lambda_n \varphi_n(\omega)\psi_n(t) \right)^2 dP d\lambda \\
&= \int_\Omega \left(\sum_{n=1}^{m} (\lambda_n \varphi_n(\omega) - \tau_n \eta_n(\omega))^2 + \sum_{n=m+1}^{\infty} \lambda_n^2 \varphi_n^2(\omega) \right) dP \\
&= \sum_{n=1}^{m} \int_\Omega (\lambda_n \varphi_n(\omega) - \tau_n \eta_n(\omega))^2 dP + \sum_{n=m+1}^{\infty} \lambda_n^2. \qquad (18)
\end{aligned}$$

(18) follows by virtue of the φ_n being orthonormal. Hence we obtain $\lambda_n \varphi_n(\omega) = \tau_n \eta_n(\omega)$ for $1 \le n \le m$. Therefore $\beta(t,\omega) = \sum_{n=1}^{m} \lambda_n \varphi_n(\omega)\psi_i(t)$, which is to say that the minimum is achieved at a unique function. ∎

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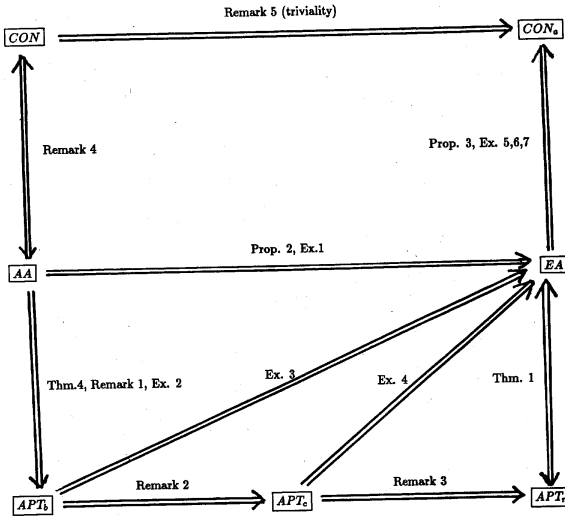

EA : A market permitting no *exact arbitrage* opportunities (Definition 4).
AA : A market permitting no *asymptotic arbitrage* opportunities (Definition 6).
APT_b : A market with APT pricing formula with bounded total square deviations (Definition 7).
APT_e : A market with exact factor-pricing for all but countably many assets (Definition 8).
APT_n : A market with exact factor-pricing for almost-all assets (Definition 8).
CON : A market in which a continuous functional prices all assets.
CON_a : A market in which a continuous functional prices almost-all assets.

Examples show that the reverse implication is false.

Figure 1: Implications in Section 3.1