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# Nonparametric analysis of random utility models: testing

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**cemmap** working paper CWP36/13

# NONPARAMETRIC ANALYSIS OF RANDOM UTILITY MODELS: TESTING

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ABSTRACT. This paper develops and implements a nonparametric test of Random Utility Models (RUM) using only nonsatiation and the Strong Axiom of Revealed Preference (SARP) as restrictions on individual level behavior, allowing for fully unrestricted unobserved heterogeneity. The main application is the test of the null hypothesis that a sample of cross-sectional demand distributions was generated by a population of rational consumers. Thus, the paper provides a finite sample counterpart to the classic theoretical analysis of McFadden and Richter (1991). To do so, it overcomes challenges in computation and in asymptotic theory and provides an empirical application to the U.K. Household Expenditure Survey. An econometric result of independent interest is a test for inequality constraints when they are represented in terms of the rays of a cone rather than its faces.

## 1. INTRODUCTION

This paper develops new tools for the analysis of Random Utility Models (RUM). The leading application is stochastic revealed preference theory, that is, the modeling of aggregate choice behavior in a population characterized by individual rationality and unobserved heterogeneity. We test the null hypothesis that a repeated cross-section of demand data was generated by such a population, without restricting unobserved heterogeneity in any form whatsoever. Equivalently, we empirically test McFadden and Richter's (1991) Axiom of Revealed Stochastic Preference (ARSP, to be defined later), using only nonsatiation and the Strong Axiom of Revealed Preference (SARP) as restrictions on individual level behavior. Doing this is computationally challenging. We provide various algorithms that can be implemented with reasonable computational resources. Also, new tools for statistical inference for inequality restrictions are introduced in order to deal with the high-dimensionality and non-regularity of the problem at hand.

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Let

$$u : \mathbf{R}_+^K \rightarrow \mathbf{R}$$

denote a random utility function. In our preferred interpretation, randomness of  $u$  represents unobserved heterogeneity across individuals.<sup>1</sup> Each consumer faces an income level and a price vector in  $\mathbf{R}_+^K$ . Normalizing income to 1, the budget set for each consumer can be denoted as  $B(p)$ ,  $p \in \mathbf{R}_+^K$ , and the consumer's choice is determined as

$$y = \arg \max_{y \in B(p)} u(y).$$

The econometrician observes a random sample of  $(y, p)$ . In other words, she observes (a sample analog of) choice probability

$$\Pr(y \in Y \mid \text{price is } p)$$

for each  $Y \subset \mathbf{R}_+^K$ . The question is whether (up to sampling uncertainty) these choice probabilities can be rationalized as an outcome of RUM. Our approach can be briefly described by the following steps:

- The first insight is that, although demand data are continuous, they can be discretized without any loss of information as long as the set of budgets is finite. Thus, a random utility model of demand on a finite set of budgets is really a model of discrete choice, though the number of distinct choice objects can be large (up to 67 in our empirical application in Section 8). The next steps of our approach immediately apply to choice problems that were discrete to begin with.
- If there is a finite list of discrete choice problems, then there is a finite list of rational “choice types.” Each such type is uniquely characterized by a rationalizable nonstochastic choice pattern.<sup>2</sup> In realistic problem sizes, there are many such types (up to 177352 in our application), and obtaining the list is computationally challenging. Some techniques for efficiently computing the list are an important part of our contribution.
- Think of every rational choice type as defining a vector of degenerate choice probabilities over discrete choice objects. Then a corresponding vector of nondegenerate choice probabilities is

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<sup>1</sup>Random utility models were originally developed in mathematical psychology, and in principle, our results also apply to stochastic choice behavior by an individual. However, in these settings it would frequently be natural to impose much more structure than we do.

<sup>2</sup>Note that we do not restrict the distribution of the utility function  $u$  to be discrete. It is the fundamental nature of our problem that enables us to convert it into a discrete problem, albeit a very high dimensional one.

consistent with a random utility model iff it is a convex combination of the degenerate ones. We collect the latter into columns of a matrix; choice probabilities are then rationalizable iff they are in the column cone spanned by this matrix. The same insight informs our test statistic, which will be weighted Euclidean distance of estimated choice probabilities from this cone. In particular, it is computationally convenient to work with this cone and not the polytope that is generated by explicitly imposing that choice probabilities must be proper probabilities.

- The limiting distribution of the test statistic depends discontinuously on a high-dimensional nuisance parameter. Such features have been studied extensively in the literature on moment inequalities. However, our problem has characteristics which are not well studied in that literature. In particular, we will effectively test moment inequalities that jointly define a convex cone, and this cone will be represented in terms of its spanning rays rather than its faces. Moving from one representation to the other is computationally infeasible. We therefore use the representation of the cone that emerges naturally in our problem and “tighten” it for the purpose of bootstrap simulations. The procedure is similar in spirit to approaches proposed by Andrews and Soares (2010), Bugni (2010), Canay (2010), and others. However, it is easy to implement in our application and, more generally, will have practical appeal whenever the representation of inequalities through a cone’s spanning rays is natural.

In summary, the present paper contributes to the literature by (i) showing that McFadden and Richter’s ARSP, and in particular stochastic rationality of a population, can be tested nonparametrically, and developing computational tools to do so, and (ii) proposing a method based on “inequality tightening” that works well for inequality testing in some settings where standard methods as in Andrews and Soares (2010) or Bugni (2010) are hard to implement. We note that (ii) is also a contribution to specification testing of partially identified moment inequalities models more generally, namely in cases where the indirect moment inequalities as discussed here are the natural description of a model. We will also briefly explain how to carry out counterfactual analysis and intend to flesh out this part of the analysis in a companion paper.

## 2. RELATED LITERATURE

In this section, we discuss the related literature on demand; related works in asymptotic theory are more easily discussed after setting up the corresponding framework. We first and foremost build on the classic literature on (deterministic and stochastic) revealed preference. Inspired by Samuelson’s

(1938) statement of the revealed preference paradigm, Houthakker (1950), Afriat (1967; see also Varian (1982)), and Richter (1966) delineated the precise content of utility maximization if one observes a single consumer’s demand behavior. For our purposes, this content is embodied in the Strong Axiom of Revealed Preference (SARP): The transitive closure of directly revealed preference must be acyclical.<sup>3</sup> This approach was extended to random utility maximization by Block and Marschak (1960), Falmagne (1978), McFadden and Richter (1991), and McFadden (2005). In particular, McFadden and Richter show that the precise content of random utility maximization – or equivalently, of individual level utility maximization in the presence of unrestricted, unobservable heterogeneity – is expressed by a collection of inequalities collectively dubbed “Axiom of Revealed Stochastic Preference” (ARSP).

These findings resolve this paper’s questions “in the limit” when all identifiable quantities are known. In this idealized setting, they allow one to decide with certainty whether a given demand system or distribution of demands is rationalizable. In reality, estimators of these quantities might fail to be rationalizable because of sampling variation, and one can merely test the hypothesis that data might have been generated by a rational individual or population. For testing individual level rationality, such a test was proposed by Epstein and Yatchew (1985).<sup>4</sup> To the best of our knowledge, we provide the first such test for ARSP.

Perhaps the closest paper to ours in spirit is Manski (2007). In a simple, very abstract discrete choice problem (the universal set of options is a finite and, in practice, small list), he analyzes essentially the same question as we do. In particular, he states the testing and extrapolation problems in the abstract, solves them in simple examples, and outlines an approach to exact finite sample inference. Further results for simple instances of the problem, including results on the degree of underidentification of choice types, were provided by Sher et al. (2011). While we start from a continuous problem

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<sup>3</sup>The afocited papers subtly differ in their handling of indifference. SARP characterizes rationality in the absence of indifference; else, it is sufficient but not necessary. Richter (1966) characterizes rationality if indifference is revealed through set-valued choice. The Afriat inequalities, or equivalently Varian’s (1982) Generalized Axiom of Revealed Preference (GARP), characterize rationality if indifference is permitted but cannot be revealed through set-valued choice.

These differences do not matter in our setting. Observed choice is always unique, hence Richter’s (1966) axiom collapses to SARP. Because nonsatiation will be assumed, GARP differs from SARP only with respect to choice objects that lie on the intersections of budget planes. With continuous demand on finitely many budgets, this case has zero probability (and does not occur in our data). In cases where these subtleties do matter, it would be easy to adopt our approach to any of these variations.

<sup>4</sup>See also the survey by Cherchye et al. (2009) and recent work by Dean and Martin (2013) or Echenique et al. (2011) for other approaches to testing individual level rationality.

and use asymptotic theory rather than exact inference, the settings become similar after our initial discretization step. Our main contribution relative to these papers is to provide the computational toolkit, as well as asymptotic theory, to handle problems of realistic size. Indeed, this toolkit was recently employed for choice extrapolation by Manski (2013).

In a series of influential papers, Blundell, Browning, and Crawford (2003, 2008; BBC henceforth) develop a nonparametric approach to demand analysis based on Engel curves. They assume the same observables as we do and apply their method to the British Family Expenditure Survey (FES), to which we apply our method as well in Section 8. The core difference to our approach is that BBC analyze one individual level demand system generated by nonparametric estimation of Engel curves. This could be loosely characterized as revealed preference analysis of a representative consumer (and in practice, given their specific estimation technique, of average demand). One possible foundation for it was provided by Lewbel (2001), who gives conditions on the random part of a random utility model that ensure integrability of expected demand. Lewbel's (2001) assumptions therefore precisely delineate one possible bridge between BBC's assumptions and ours. Also, BBC exploit only the implications of the Weak Axiom of Revealed Preference (WARP) and therefore test a necessary but not sufficient condition for rationalizability of aggregate demand; see Blundell, Browning, Cherchye, Crawford, de Rock, and Vermeulen (2012). In contrast, we consider the full implication of SARP. A different test of necessary conditions is proposed by Hoderlein (2011), who shows that certain features of rational individual demand, like adding up and standard properties of the Slutsky matrix, are inherited by aggregate demand under weak conditions. The resulting test is passed by the FES data.

Hoderlein and Stoye (2013) again use the same assumptions and data but ask different questions. They bound from above and below the fraction of the population who violate WARP. As a corollary (namely, by exhibiting the conditions under which the lower bound is zero), they show what discipline is put on the data by WARP alone. In the very special case of two goods, their results are formally connected to ARSP because WARP implies SARP (Rose (1958)), thus their corollary and ARSP must have the same specialization. This specialization is developed, and some implications are pointed out, in Stoye and Hoderlein (2013). The latter paper contains no asymptotic theory, and the asymptotic theory in Hoderlein and Stoye (2013) is more closely related to the previous literature on moment inequalities than to this paper's innovation.

Finally, our approach can be usefully contrasted to the recently active literature on invertibility of demand, that is, on conditions under which individual demand can be backed out from observation of repeated cross-sections. See Beckert and Blundell (2007), Berry, Gandhi, and Haile (2013), and

references therein. Unsurprisingly, invertibility requires substantial assumptions on structural parameters, i.e. utility functions and/or their distributions, which we avoid. The paper in this literature that is perhaps closest to ours is Blundell, Kristensen, and Matzkin (2011), who investigate nonparametric extrapolation of demand and, compared to our setting, essentially add only invertibility. Other than by adding this assumption, their paper differs from ours by restricting attention to two goods. The extension of their approach to more than two goods is challenging; however, in the case of two goods, Stoye and Hoderlein (2013) show that their invertibility assumption is without loss of generality given the data structure.

### 3. METHODOLOGY

Following McFadden and Richter (1991) as well as many of the aforementioned references, we presume a finite number of budgets parameterized by  $p \in \{p_1, \dots, p_J\}$ .<sup>5</sup> Indexing the corresponding budget planes as  $B_j = B(p_j) := \{y \geq 0 : p'_j y = 1\}$ , we can drop  $p$  from notation and write choice probability functions as

$$\pi(y \in Y|B_j) := \Pr(y \in Y|\text{budget plane is } B_j), \quad Y \subset \mathcal{Y}.$$

We only restrict individual consumers' behavior by monotonicity ("more is better") and SARP; equivalently, we assume that each consumer maximizes some nondecreasing utility function.<sup>6</sup> The only implication of monotonicity in our setting is that choices must be on (and not below) budget planes. Thus, we assume that demand distributions are supported on the corresponding budget planes; more formally,  $\pi(y \in B_j|B_j) = 1$ . The only implication of SARP is to restrict, in possibly very intricate ways, whether choice can simultaneously be above certain budgets and below others. In particular, consider two consumers  $a$  and  $b$  whose choices  $y_j^a$  and  $y_j^b$  are different but fulfil  $(p'_j y_k^a - 1)(p'_j y_k^b - 1) \geq 0$  for each  $(j, k) \in \{1, \dots, J\}^2$ . In words, it is true for any  $(j, k)$  that consumer  $a$ 's choice from budget  $j$  lies above budget  $k$  iff consumer  $b$ 's choice from budget  $j$  does. Then it must be the case that either both consumers fulfil SARP or both violate it. For the purpose of testing our random utility model, we can take the two consumers to form an equivalence class.

This insight means that, while the space of choices  $\mathcal{Y} \subset \mathbf{R}_+^K$  is generally unrestricted and is continuous in our application, it can be discretized without loss of generality. Doing so requires some

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<sup>5</sup>The theory was extended to the continuous case in McFadden (2005), and we discuss prospects for the corresponding extension of our approach in the conclusion.

<sup>6</sup>In a setting characterized by choice from linear budgets, this is furthermore equivalent to assuming maximization of strictly concave utility (Varian (1982)).



definitions. For every budget  $B_j$ , let the  $I_j$  elements of  $\{x_{1|j}, \dots, x_{I_j|j}\}$  form the coarsest partition of  $B_j$  such that no budget plane other than  $B_j$  intersects the interior of any one element of the partition. In other words,  $\{x_{1|j}, \dots, x_{I_j|j}\}$  has the property that its elements are disjoint, that  $B_j = \bigcup_{i=1}^{I_j} x_{i|j}$  for every  $j$ , and also that for any  $k = 1, \dots, J$ , any  $i = 1, \dots, I_j$ , and any  $y_1, y_2 \in x_{i|j}$ , one has  $(p'_k y_1 - 1)(p'_k y_2 - 1) \geq 0$ ; furthermore,  $\{x_{1|j}, \dots, x_{I_j|j}\}$  is the coarsest such partition. Henceforth, we use the word “patch” to denote a generic element  $x_{i|j}$  of  $\mathcal{X}$ . For the simplest nontrivial example, let there be  $J = 2$  budgets that intersect, then there is a total of four patches: two on  $B_1$ , where one is above  $B_2$  and the other one is below it, and two patches on  $B_2$ , where one is above  $B_1$  and one is below it. Then the random utility model intricately restricts the probabilities of patches  $\pi(y \in x_{i|j} | B_j)$ , but it does not at all restrict the conditional distributions of demand on patches.

We can, therefore, work with choice space  $\mathcal{X}$ , effectively discretizing the choice problem to one with a total of  $I := \sum_{j=1}^J I_j$  choice objects and corresponding choice probabilities. This leaves us with a large (in applications of practical interest) but finite set of distinct nonstochastic choice types. We will now explain how to efficiently encode these. To do so, arrange  $\mathcal{X}$  as a vector

$$\mathcal{X} = (x_{1|1}, x_{2|1}, \dots, x_{I_J|J})'$$

and similarly write

$$\begin{aligned} \pi_{i|j} &:= \Pr(y \in x_{i|j} | B_j), \\ \pi_j &:= (\pi_{1|j}, \dots, \pi_{I_j|j})' \\ \pi &:= (\pi'_1, \dots, \pi'_J)' = (\pi_{1|1}, \pi_{2|1}, \dots, \pi_{I_J|J})', \end{aligned}$$

then the  $I$ -vector  $\pi$  contains all information that is relevant for testing RUM.

Any conceivable pattern of nonstochastic choice behavior can be identified with a binary  $I$ -vector  $a = (a_{1|1}, \dots, a_{I_J|J})'$ , where a component  $a_{i|j}$  of  $a$  equals 1 iff the corresponding component  $x_{i|j}$  of  $\mathcal{X}$  is chosen from budget  $B_j$ . We call  $a$  rationalizable if there exists  $u^* \in \mathcal{U}$  such that  $a_{i|j} = \mathbf{1}\{\arg \max_{y \in B_j} u^*(x) \in x_{i|j}\}$  for all  $a_{i|j}$ . Under our assumptions, this is the case iff behavior encoded in  $a$  fulfills SARP, but we emphasize that the approach could be modified to impose more or less structure on  $a$  and thereby on agents' behavior. Every rationalizable vector  $a$  can be thought of as characterizing a rational, nonstochastic choice type. Let the columns of the  $(I \times H)$ -matrix  $A := [a_1, \dots, a_H]$  collect all rationalizable such vectors. Then a vector of choice probabilities  $\pi$  is stochastically rational iff there exists  $\nu \in \Delta^{H-1}$  such that  $A\nu = \pi$ . In words,  $\pi$  must be a convex combination of the columns of  $A$ . The weights  $\nu$  can be interpreted as implied population distribution over rational choice types;

remember, though, that each of these choice types represents an equivalence class of observationally distinct choice types.

To compare our assumptions with those inherent in other approaches, observe how invertibility of demand fails on many levels. First, we consider finitely many budgets, so with unconstrained heterogeneity, any given demand pattern is rationalizable iff it is rationalizable by infinitely many utility functions. Second, every column  $a$  represents a continuum of observationally distinct, rationalizable demand patterns because of the lumping of continuous demand data into patches. Third,  $\nu$  is not identified:  $A$  is typically very far from full column rank, and a  $\pi$  that is rationalizable at all will typically be rationalizable by many different vectors  $\nu$ . We return to partial identification of  $\nu$  in section 9.1.

McFadden and Richter (1991; see also McFadden (2005, theorem 3.3)) anticipated the discretization of choice space and also noted various equivalent statements for the empirical content of RUM. For example,  $\nu$  as required here exists iff the linear program  $\min_{\nu \geq 0, s \geq 0} \mathbf{1}'_1 s$  s.t.  $A\nu + s \geq \pi, \mathbf{1}'_H \nu \leq 1$  has an optimal solution with  $s = 0$ . However, we employ the first statement verbalized above, thus we directly test:

**(H<sub>A</sub>):** There exist a  $\nu \in \Delta^{H-1}$  such that  $A\nu = \pi$ ,

where  $\Delta^{H-1}$  denotes the  $(H-1)$ -dimensional unit simplex. To test this hypothesis, we transform it as follows. First, note that  $\mathbf{1}'_J A = [J, \dots, J]$  and  $\mathbf{1}'_J \pi = J$  hold by definition. Therefore,  $A\nu = \pi$  implies  $\mathbf{1}'_H \nu = 1$ . Thus **(H<sub>A</sub>)** is equivalent to

**(H<sub>B</sub>):** There exist  $\nu \geq 0$  such that  $A\nu = \pi$ .

It is easy to see that Hypothesis **(H<sub>B</sub>)** is, in turn, equivalent to

**(H<sub>C</sub>):**  $\min_{\eta \in C} [\pi - \eta]' \Omega [\pi - \eta] = 0$ ,

where  $\Omega$  is a positive definite matrix (restricted to be diagonal in our inference procedure) and  $C := \{A\nu | \nu \geq 0\}$ . Note that the constraint set  $C$  is a cone in  $\mathbf{R}^I$ . The solution  $\eta_0$  of **(H<sub>C</sub>)** is the projection of  $\pi \in \mathbf{R}^I_+$  onto  $C$  under the weighted norm  $\|x\|_\Omega = \sqrt{x' \Omega x}$ . The corresponding value of the objective function is the squared length of the projection residual vector. The projection  $\eta_0$  is unique, but the

corresponding  $\nu$  is not. Stochastic rationality holds if and only if the length of the residual vector is zero.

A natural sample counterpart of the objective function in  $(\mathbf{H}_C)$  would be  $\min_{\eta \in C} [\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta]$ , where  $\hat{\pi}$  estimates  $\pi$ , for example by sample choice frequencies. It is useful to normalize this sample counterpart by  $N$  to obtain an appropriate asymptotic distribution, so define

$$(3.1) \quad \begin{aligned} J_N & : = N \min_{\eta \in C} [\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta] \\ & = N \min_{\nu \in \mathbf{R}_+^H} [\hat{\pi} - A\nu]' \Omega [\hat{\pi} - A\nu]. \end{aligned}$$

Once again,  $\nu$  is not unique at the optimum, but  $\eta = A\nu$  is. Call its optimal value  $\hat{\eta}$ , noting that  $\hat{\eta}$  can also be thought of as rationality-constrained estimator of choice probabilities. Then  $\hat{\eta} = \hat{\pi}$ , and  $J_N = 0$ , iff the estimated choice probabilities  $\hat{\pi}$  are stochastically rational; obviously, our null hypothesis will be accepted in this case. To determine an appropriate critical value for our test, we will have to estimate the distribution of  $J_N$ . This estimation problem is intricate and will be handled in section 5.

#### 4. COMPUTATION

We now turn to computational implementation of the approach. The challenge is threefold: First, how to encode the choice set  $\mathcal{X}$ ; second, how to generate the matrix  $A$ ; last, how to carry out the constrained minimization in the definition of  $J_N$ . We explain our response to these issues in this order.

**4.1. Encoding  $\mathcal{X}$ .** Geometrically, the patches  $x_{i|j}$  are polyhedra with rather complex descriptions. However, the only information that matters for computing  $A$  is how any given patch relates to the different budgets. To implement this insight, we switch from the double index expression  $\mathcal{X} = \{\{x_{i|j}\}_{i=1}^{I_j}\}_{j=1}^J$  to a single index  $i = 1, \dots, I$  and write  $\mathcal{X} = \{x_i\}_{i=1}^I$  in the remainder of this section. Now, let

$$X_{ij} := \begin{cases} -1 & \text{if } x_i \text{ is below } B_j \\ 0 & \text{if } x_i \text{ is on } B_j \\ +1 & \text{if } x_i \text{ is above } B_j \end{cases} \quad j = 1, \dots, J,$$

then we can represent each  $x_i$  by the vector

$$X_i := [X_{i1}, \dots, X_{iJ}], \quad i = 1, \dots, I.$$

The choice set is represented by the  $(I \times J)$ -matrix  $X = [X'_1, \dots, X'_J]'$  with typical cell  $X_{ij}$ . Our algorithm for generating  $X$  from budget data  $(p_1, \dots, p_J)$  is as follows: First, generate all  $J \times 2^{J-1}$  possible vectors  $X_i$  of the form just described. Not all of these will encode patches that actually exist in the given choice problem; in particular, some of them will define intersections of half-spaces that do not occur in the positive quadrant. Thus, test the validity of each vector and append it to  $X$  only if the test is passed. To illustrate this step, suppose that  $J = 5$  with  $K$  goods and that we want to verify whether the patch encoded by  $(0, -1, 1, 1, 1)$  exists. It is easy to see that this is the case iff the system of  $(J + 1 + K)$  inequalities

$$p'_1 y \leq 1, p'_1 y \geq 1, p'_2 y \leq 1, p'_3 y \geq 1, p'_4 y \geq 1, p'_5 y \geq 1, y \geq 0$$

has a solution. Several numerical solvers can quickly verify consistency of a set of linear inequality constraints. Using the **CVX** optimization package (Grant and Boyd (2008, 2011)), we find that checking the above inequalities is computationally inexpensive even for high dimensional commodity spaces.<sup>7</sup>

**4.2. Computing  $A$ .** To compute the matrix  $A$ , write

$$A = \{a_{ih}\},$$

where  $a_{ih}$  equals 1 if choice type  $h$  picks patch  $x_i$  from budget  $B_j$ , where  $B_j$  contains  $x_i$ . (As with  $\mathcal{X}$ , it is convenient for this section's purpose to index rows of  $A$  by  $i = 1, \dots, I$  instead of the double index  $i|j$ .) The challenge is to ensure that a nonstochastic choice type is represented as a column of  $A$  iff her choices are rationalizable, i.e. iff the (transitive closure of the) incomplete preference ordering revealed by her choice behavior is acyclical. To do this, we first extract the preferences revealed by a choice type, then check for acyclicity. The following fact is crucial for the first step: Choice of  $x_i$  from budget  $B_j$  reveals that  $x_i \succ x_m$  for every patch  $x_m \neq x_j$  s.t.  $X_{mj} \in \{-1, 0\}$  (the symbol  $\succ$  signifies the revealed preference relation). Here,  $X_{mj} = 0 \Rightarrow x_i \succ x_m$  follows directly because  $x_i$  and  $x_m$  are on the same budget plane;  $X_{mj} = -1 \Rightarrow x_i \succ x_m$  follows by additional uses of "more is better" and transitivity because  $x_i$  was chosen over some patch that dominates  $x_m$ . In a second step, acyclicity of a given set of revealed preferences can be checked by the Floyd-Warshall algorithm, efficient implementations of which are readily available.

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<sup>7</sup>If this computation were a bottleneck, it could be refined along the lines of the decision tree crawling algorithm discussed in the next subsection. However, computation of  $A$  is the much harder step.

We implemented three ways to generate  $A$ .<sup>8</sup> First, a “brute force” approach initially creates a  $\left(I \times \prod_{j=1}^J I_j\right)$ -matrix  $A^{\max}$  that represents all possible choice combinations from  $J$  budgets. Every column of  $A^{\max}$  is then checked for rationality. This approach, which resembles our computation of  $X$ , has the benefit of being straightforward and of also collecting irrational types, which can be handy for simulations. However, the number of columns of  $A^{\max}$  grows extremely rapidly as budgets are added, and the brute force approach is not practical in realistic examples, including this paper’s empirical application.

Our second approach is a “decision tree crawling” algorithm. Here, the core insight is that all possible choice patterns can be associated with terminal nodes of a decision tree whose initial node corresponds to choice from  $B_1$ , the next set of nodes to choice from  $B_2$ , and so on. The algorithm exhaustively crawls this tree, checking for choice cycles at every node that is visited and abandoning the entire branch whenever a cycle is detected. A column of  $A$  is discovered every time that a terminal node is visited without detecting a cycle. The abandoning of branches means that most nonrationalizable choice patterns are never visited. For example, if a cycle is detected after specifying behavior on 4 out of 10 budgets, then none of the many possible completions of this choice pattern are considered. The downside is more frequent (for any rational pattern that is detected) execution of the Floyd-Warshall algorithm, but this algorithm is cheap (it terminates in polynomial time). The net effect is to improve computation time by orders of magnitude in complicated problems. Indeed, this is our most powerful algorithm that is directly applicable “off the shelf” in every application of our approach.

Finally, a modest amount of problem-specific adjustment can lead to further, dramatic improvement in many cases, including our empirical application. The key to this is contained in the following proposition, which is established in appendix A.

**Proposition 4.1.** *Suppose that for some  $M \geq 1$ , none of  $(B_1, \dots, B_M)$  intersect  $B_J$ . Suppose also that choices from  $(B_1, \dots, B_{J-1})$  are jointly rationalizable. Then choices from  $(B_1, \dots, B_J)$  are jointly rationalizable iff choices from  $(B_{M+1}, \dots, B_J)$  are.*

This proposition is helpful whenever not all budgets mutually intersect. To exploit it in applications, one must manually check for such sets of budgets and possibly be willing to reorder budgets. The benefit is that if the proposition applies, all rationalizable choice patterns can be discovered by

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<sup>8</sup>Appendix B contains a more detailed description, including pseudo-code for some algorithms.

checking for rationalizable choice patterns on smaller domains and then combining the results. In particular, Proposition 4.1 informs the following strategy: First collect all rationalizable choice patterns on  $(B_{M+1}, \dots, B_{J-1})$ . Next, for each such pattern, find all rationalizable completions of it to choice on  $(B_1, \dots, B_{J-1})$  as well as all rationalizable completions to  $(B_{M+1}, \dots, B_J)$ . (For the first of these steps, one may further economize by computing all rationalizable patterns on  $(B_1, \dots, B_M)$  in a preliminary step.) Every combination of two such completions is itself a rationalizable choice pattern. Note that no step in this algorithm checks rationality on  $J$  budgets at once; furthermore, a Cartesian product structure of the set of rationalizable choice patterns is exploited. The potential benefit is substantial – in our application, the refinement sometimes improves computation times by orders of magnitude.

**4.3. Computing  $J_N$ .** Computation of  $J_N$  is a quadratic programming problem subject to a possibly large number of linear inequality constraints. We have nothing to add to the theory of solving such problems and rely on modern numerical solvers that can handle high-dimensional quadratic programming problems. Our currently preferred implementation utilizes **CVX**. We also implemented computation of  $J_N$  with **fmincon** (using stepwise quadratic programming) and **Knitro**. All solvers agree on those problems that they can handle, with **fmincon** being practical only for rather small problem sizes.

**4.4. Summary.** As we will demonstrate in Section 8, the above algorithms for computation of  $X$ ,  $A$ , and  $J_N$  can be executed for high-dimensional problems. However, we will now illustrate by very briefly going over the simplest nontrivial example, i.e. two budgets which intersect. In this case  $J = 2$  and  $I_1 = I_2 = 2$ , yielding  $I = 4$  patches, and we have

$$X = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \pi = \begin{bmatrix} \Pr(x_1 \text{ is chosen} | \text{budget is } B_1) \\ \Pr(x_2 \text{ is chosen} | \text{budget is } B_1) \\ \Pr(x_3 \text{ is chosen} | \text{budget is } B_2) \\ \Pr(x_4 \text{ is chosen} | \text{budget is } B_2) \end{bmatrix}.$$

Here, the first row of  $X$  represents the part of budget  $B_1$  that is below  $B_2$  and so on. Rows of  $A$  corresponds to rows of  $X$ , whereas its columns correspond to types. Thus, the first column of  $A$  indicates the rational choice type whose choice from  $B_1$  is below  $B_2$  but whose choice from  $B_2$  is above  $B_1$ . There are four logically possible choice types, but one of them, namely  $[1, 0, 1, 0]'$ , would violate WARP and is therefore not represented in  $A$ . Given an estimator  $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)'$  for  $\pi$  and setting  $\Omega = I_2$ , one can then verify that  $J_N = N \cdot (\max\{\hat{\pi}_1 + \hat{\pi}_3 - 1, 0\})^2$ . In particular, the

test statistic  $J_N$  is zero if  $\hat{\pi}_1 + \hat{\pi}_3 \leq 1$ , and in that case we immediately conclude that the data are consistent with the random utility model.

The example reproduces the known finding (Matzkin (2006), Hoderlein and Stoye (2013)) that with two budgets, the content of stochastic rationality is exhausted by the restriction “ $\pi_1 + \pi_3 \leq 1$ ” on population choice probabilities. However, the tools proposed here allow one to perform similar computations for very complicated examples. Think of the size of  $A$ , that is, “number of patches  $\times$  number of rational choice types,” as an indicator of problem complexity. Then the example just given has size  $(4 \times 3)$ . For our empirical application, we successfully computed  $A$ , and computed  $J_N$  2000 times, in cases where  $A$  has size  $(67 \times 149570)$  respectively  $(64 \times 177352)$ . The computational bottlenecks in implementation are twofold: Computation of  $A$  took several hours in some cases (although reduced to minutes using proposition 4.1), and computation of  $J_N$  (which must be iterated over to compute critical values) took up to a minute.

## 5. INFERENCE

This section discusses inferential procedures to deal with the hypothesis ( $\mathbf{H}_A$ ) or its equivalent forms. We initially assume that choice probabilities were estimated by sample frequencies. For each budget  $j$ , denote the choices of  $N_j$  individuals, indexed by  $n = 1, \dots, N_j$ , from the budget set  $B_j$ , by

$$d_{i|j,n} = \begin{cases} 1 & \text{if individual } n \text{ chooses } x_{i|j} \\ 0 & \text{otherwise} \end{cases} \quad n = 1, \dots, N_j.$$

Assume that one observes  $J$  random samples  $\{\{d_{i|j,n}\}_{i=1}^{I_j}\}_{n=1}^{N_j}$ ,  $j = 1, 2, \dots, J$ . For later use, define

$$d_{j,n} := \begin{bmatrix} d_{1|j,n} \\ \vdots \\ d_{I_j|j,n} \end{bmatrix}, \quad N = \sum_{j=1}^J N_j.$$

An obvious way to estimate the vector  $\pi$  is to use choice frequencies

$$(5.1) \quad \hat{\pi}_{i|j} = \sum_{n=1}^{N_j} d_{i|j,n} / N_j, \quad i = 1, \dots, I_j, \quad j = 1, \dots, J.$$

Our main task is to estimate the sampling distribution of  $J_N = N \min_{\nu \in \mathbf{R}_+^H} [\hat{\pi} - A\nu]' \Omega [\hat{\pi} - A\nu]$  under the null hypothesis that the true  $\pi$  is rationalizable. This problem is closely related to the literature on inequality testing, but with an important twist. Recall that  $\{a_1, a_2, \dots, a_H\}$  are the column vectors

of  $A$ . Define

$$\text{cone}(A) = \{\nu_1 a_1 + \dots + \nu_H a_H : \nu_h \geq 0\},$$

then  $C = \text{cone}(A)$ , and we test whether  $\pi \in C$ . We therefore work with what is called a  $\mathcal{V}$ -representation of the cone  $C$  in the literature on convex geometry (see, e.g., Ziegler (1995)). Unfortunately, a  $\mathcal{V}$ -representation is not useful in detecting whether  $\hat{\eta}$  is close to an irregular point or not. (Irregular points occur close to the spanning rays of the cone, but because many vectors  $\nu$  correspond to the same projection  $\hat{\eta}$ , we may not be able to diagnose this from the  $\nu$  that we discovered.) Thus, an alternative description of  $C$  in terms of  $\eta$  has to be obtained. Weyl's Theorem guarantees that that is possible *theoretically*: If a set  $C$  in  $\mathbf{R}^J$  is represented as  $\text{cone}(A)$ ,  $A \in \mathbf{R}^{I \times H}$ , we can write

$$C = \{t \in \mathbf{R}^I \mid Bt \leq 0\} \text{ for some } B \in \mathbf{R}^{m \times I}.$$

The last line is called an  $\mathcal{H}$ -representation of  $C$ .<sup>9</sup> Applying Weyl to the definition of  $J_N$ , one can obtain

$$J_N = \min_{t \in \mathbf{R}^I : Bt \leq 0} N[\hat{\pi} - t]' \Omega [\hat{\pi} - t],$$

where the optimal  $t$  ( $:= \hat{t}$ ) is unique.

On a theoretical level, this connects our testing problem to tests of

$$H_0 : B\theta \geq 0 \quad B \in \mathbf{R}^{p \times q} \text{ is known}$$

based on test statistics of form

$$T_N := \min_{\eta \in \mathbf{R}_+^q} N[B\hat{\theta} - \eta]' S^{-1} [B\hat{\theta} - \eta].$$

This type of problem has been studied by Gourieroux, Holly and Monfort (1982) and Wolak (1991). See also Chernoff (1954), Kudo (1963), Perlman (1969), Shapiro (1988), Kuriki and Takemura (2000), Andrews (2001), Rosen (2008), and Guggenberger, Hahn, and Kim (2008).<sup>10</sup> A common way to get a critical value for  $T_N$  is to consider the least favorable case, which is  $\theta = 0$ . This strategy is

<sup>9</sup>The converse is also true and known as Minkowski's Theorem. See Gruber (2007), Grünbaum (2003) and Ziegler (1995) for these results and other materials concerning convex polytopes used in this paper.

<sup>10</sup>More precisely, Guggenberger, Hahn and Kim (2008) consider specification testing of linear moment inequality models of the form  $C\theta \leq \mathbb{E}[x]$ ,  $\theta \in \mathbf{R}^m$  where  $C$  is a known conformable matrix, and propose to test its equivalent form  $R\mathbb{E}[x] \geq 0$  where  $R$  is another known matrix. This equivalence follows from the Weyl-Minkowski Theorem, which we use as well. Moving from  $C$  to  $R$  is not computationally feasible in our application, however, and we avoid it. Similarly, Canay's (2010) inequality tightening is related to our method, but applying it here would require the  $\mathcal{H}$ -representation of  $C$ , which is not in practice available.



inappropriate in the present context for a number of reasons: Direct application to our problem is not feasible because the  $\mathcal{H}$ -representation of  $C$  cannot be computed in practice, and the suggested least favorable distribution can lead to unnecessarily conservative inference. Alternatively, one might consider a resampling method. However, some care is required in calculating a valid critical value for  $J_N$  in cases where more than one inequality is (almost) binding. The naive bootstrap and like methods fail in these cases. (See, for example, Andrews and Soares (2010), Bugni (2010), Canay (2010), Chernozhukov, Hong and Tamer (2007), Imbens and Manski (2004), Romano and Shaikh (2010), and Stoye (2009).)

Three methods for inference that can be potentially used to calculate a critical value for the statistic  $J_N$  are: (1) Regularization, (2) moment selection, and (3) “inequality tightening.” We now discuss them in turn. Suppose that  $\sqrt{N}(\hat{\pi} - \pi) \rightarrow_d N(0, S)$ , where  $S$  is the asymptotic covariance matrix. Let  $\hat{S}$  denote a consistent estimator for  $S$ . Then regularization is easy to implement. Let  $\tilde{\eta}_{\alpha_N} := \hat{\eta} + \sqrt{\frac{\alpha_N}{N}}N(0, \hat{S})$ , where  $\alpha_N$  is a sequence that goes to infinity slowly. Recall that  $\hat{\eta}$  is the projection of the choice frequency vector  $\hat{\pi}$  onto the cone  $C$ . The random variable  $\tilde{\eta}_{\alpha_N}$  is essentially a subsampled or  $m$ -out-of- $n$  bootstrapped version of  $\hat{\eta}$ . The distribution of  $\tilde{J}_N(\alpha_N) := \frac{N}{\alpha_N} \min_{\nu \in \mathbf{R}_+^H} [\tilde{\eta}_{\alpha_N} - A\nu]' \Omega [\tilde{\eta}_{\alpha_N} - A\nu]$  can be evaluated by simulation. It provides a valid approximation of the distribution of  $J_N$  asymptotically, regardless of the position of  $\eta_0$ , the population analog of  $\hat{\eta}$ , on the cone  $C$ . This is basically the idea behind subsampling and the  $m$ -out-of- $n$  bootstrap. It is convenient computationally, but Andrews and Guggenberger (2009, 2010) forcefully argue that it can suffer from severe conservatism.

The second approach, i.e. inequality selection, is essentially the Generalized Moment Selection procedure for moment inequality models (see, e.g., Andrews and Soares (2010), Bugni (2010)). Let  $\kappa_N$  be a sequence that diverges slowly to infinity.<sup>11</sup> Let  $b_1, \dots, b_m$  be the row vectors of  $B$ . Suppose (w.l.o.g.) that  $-\sqrt{N}b_1\hat{t} \leq \kappa_N, \dots, -\sqrt{N}b_f\hat{t} \leq \kappa_N$  and  $-\sqrt{N}b_{f+1}\hat{t} \geq \kappa_N, \dots, -\sqrt{N}b_m\hat{t} \geq \kappa_N$ . Let  $B_1 = [b'_1, \dots, b'_f]'$  and  $B_2 = [b'_{f+1}, \dots, b'_m]'$ . Redo the above minimization, but change the constraints to

$$\min_{t \in \mathbf{R}^I: B_1 t = 0, B_2 t \leq 0} N[\hat{\pi} - t]' \Omega [\hat{\pi} - t].$$

Let  $\hat{t}_{select}$  denote the minimizer and define  $\tilde{\eta}_{\kappa_N} := \hat{t}_{select} + \frac{1}{\sqrt{N}}N(0, \hat{S})$ . The distribution of

$$\tilde{J}_n(\kappa_N) = \min_{t \in \mathbf{R}^I: B t \leq 0} N[\tilde{\eta}_{\kappa_N} - t]' \Omega [\tilde{\eta}_{\kappa_N} - t].$$

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<sup>11</sup>For example, Andrews and Soares (2010) suggest  $\kappa_N = \sqrt{\log N}$ .

offers a valid approximation to the distribution of  $J_N$ . This is expected to work well in finite samples, but is computationally infeasible in the present context. To our knowledge, implementing Weyl's Theorem to obtain an  $\mathcal{H}$ -representation out of the  $\mathcal{V}$ -representation is done by repeated application of (some variation of) the Fourier-Motzkin elimination algorithm, which is notoriously difficult when the dimension of  $\nu$  is high. This is exactly the case for the problem considered in this subsection, even for a small number of budgets.

We therefore propose an ‘‘inequality tightening’’ approach that sidesteps the need for an  $\mathcal{H}$ -representation of  $C$ , making it simple to implement and applicable to problems of a realistic size. The idea is to tighten the constraint by replacing

$$\nu \geq 0$$

with

$$\nu \geq \tau_N \mathbf{1}_H$$

for some positive scalar  $\tau_N$  that declines to zero slowly. (In principle, the vector  $\mathbf{1}_H$  could be any strictly positive  $H$ -vector, though a data based choice of such a vector is beyond the scope of the paper.) Now solve

$$\begin{aligned} J_N(\tau_N) &: = \min_{\eta \in C_{\tau_N}} N[\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta] \\ &= \min_{[\nu - \tau_N \mathbf{1}_H] \in \mathbf{R}_+^H} N[\hat{\pi} - A\nu]' \Omega [\hat{\pi} - A\nu] \end{aligned}$$

where  $C_{\tau_N} := \{A\nu | \nu \geq \tau_N \mathbf{1}_H\}$ , and let  $\hat{\eta}_{\tau_N}$  denote the solution. Our proof establishes that constraints that are almost binding at the original problem's solution (i.e., their slack is difficult to be distinguished from zero at the sample size) will be binding with zero slack after tightening. Let  $\tilde{\eta}_{\tau_N} := \hat{\eta}_{\tau_N} + \frac{1}{\sqrt{N}} N(0, \hat{S})$ . Notice that, as in the inequality selection procedure, no regularization (or subsampling/ $m$ -out-of- $n$  bootstrapping) is necessary at this stage. Finally, define

$$(5.2) \quad \begin{aligned} \tilde{J}_N(\tau_N) &: = \min_{\eta \in C_{\tau_N}} N[\tilde{\eta}_{\tau_N} - \eta]' \Omega [\tilde{\eta}_{\tau_N} - \eta] \\ &= \min_{[\nu - \tau_N \mathbf{1}_H] \in \mathbf{R}_+^H} N[\tilde{\eta}_{\tau_N} - A\nu]' \Omega [\tilde{\eta}_{\tau_N} - A\nu], \end{aligned}$$

and use its distribution to approximate that of  $J_N$ . This has the same theoretical justification as the inequality selection procedure. Unlike the latter, however, it avoids the use of an  $\mathcal{H}$ -representation, thus offering a computationally feasible empirical testing procedure for stochastic rationality. We now turn to providing a detailed justification for the method.

First, recall that we tighten the constraint set  $C$  using its  $\mathcal{V}$ -representation. The next lemma, among other things, shows that this corresponds to a strict tightening of its  $\mathcal{H}$ -representation, even though direct computation of the latter is infeasible. For a matrix  $B$ , let  $\text{col}(B)$  denote its column space.

**Lemma 5.1.** *For a  $\tau > 0$  and a matrix  $A \in \mathbf{R}^{I \times H}$ , define*

$$C_\tau := \{A\nu \mid \nu \geq \tau \mathbf{1}_H\}.$$

*Then  $C_\tau$  can be alternatively written as*

$$C_\tau = \{t : Bt \leq -\tau\phi\},$$

*where  $B \in \mathbf{R}^{m \times I}$ ,  $\phi \in \mathbf{R}_{++}^m$  and  $\phi \in \text{col}(B)$ .*

This is different from the Minkowski-Weyl theorem for polyhedra, which would provide an expression of  $C_\tau$  as the Minkowski sum of a convex hull and a non-negative hull. Lemma 5.1 is useful for proving the asymptotic validity of our procedure.

The following assumption is used for our asymptotic theory.

**Assumption 5.1.** *For all  $j = 1, \dots, J$ ,  $\frac{N_j}{N} \rightarrow \rho_j$  as  $N \rightarrow \infty$ , where  $\rho_j \in (0, 1)$ .*

Let  $b_{k,i}$ ,  $k = 1, \dots, m$ ,  $i = 1, \dots, I$  denote the  $(k, i)$  element of  $B$ , then define

$$b_k(j) = [b_{k,N_1+\dots+N_{j-1}+1}, b_{k,N_1+\dots+N_{j-1}+2}, \dots, b_{k,N_1+\dots+N_j}]'$$

for  $1 \leq j \leq J$  and  $1 \leq k \leq m$ . Consider the following requirement:

**Condition 5.1.** *For some  $\epsilon > 0$  and some  $j = 1, \dots, J$ ,  $\text{var}(b_k(j)'d_{j,n}) \geq \epsilon$  holds for all  $1 \leq k \leq m$ .*

Note that the distribution of observations is uniquely characterized by the vector  $\pi$ . Let  $\mathcal{P}$  denote the set of all  $\pi$ 's that satisfy Condition 5.1 for some (common) value of  $\epsilon$ .

**Theorem 5.1.** *Choose  $\tau_N$  so that  $\tau_N \downarrow 0$  and  $\sqrt{N}\tau_N \uparrow \infty$ . Also, let  $\Omega$  be diagonal, where all the diagonal elements are positive. Then under Assumption 5.1*

$$\liminf_{N \rightarrow \infty} \inf_{\pi \in \mathcal{P} \cap C} \Pr\{J_N \leq \hat{c}_{1-\alpha}\} = 1 - \alpha$$

*where  $\hat{c}_{1-\alpha}$  is the  $1 - \alpha$  quantile of  $\tilde{J}_N(\tau_N)$ ,  $0 \leq \alpha \leq \frac{1}{2}$ .*

*Proof:* See Appendix A.

Technically, the boundedness of our observations  $d_{j,n}$  — each element of  $d_{i|j,n}$  is a Bernoulli( $\pi_{i|j}$ ) random variable — and the above condition guarantee that the Lindeberg condition holds for  $b'_k \hat{\pi}$ ,  $k = 1, \dots, m$ , which is important for the uniform size control result in the following theorem as it relies on a triangular array CLT. It is known that a triangular array  $X_{iN}$ ,  $i = 1, \dots, N \sim_{\text{iid}} \text{Bernoulli}(p_N)$ ,  $N = 1, 2, \dots$  obeys CLT iff  $Np_N(1 - p_N) \rightarrow \infty$ . Since the size control here is defined against a fixed class of distribution  $\mathcal{F} \cap C$  as  $N \rightarrow \infty$ , it seems natural to impose a fixed lower bound  $\epsilon$  in Condition 5.1. See the proof of Theorem 5.1 in Appendix A for more on this point.

Finally, we note that the method works because tightening the cone can only turn non-binding inequalities from the  $\mathcal{H}$ -representation into binding ones but not vice versa. This feature is not universal to cones. Our proof establishes that it generally obtains if  $\Omega$  is the identity matrix and all corners of the cone are acute. In this paper's application, we can further exploit the cone's geometry to extend the result to any diagonal  $\Omega$ . Our method immediately applies to other  $\mathcal{V}$ -representations if analogous features can be verified.

## 6. BOOTSTRAP ALGORITHM WITH TIGHTENING

This section details how to simulate the distribution of  $J_N$  with a bootstrap procedure that employs Theorem 5.1. First, we apply the standard nonparametric bootstrap to obtain resampled unrestricted choice probability vector estimates  $\hat{\pi}^{*(r)}$ ,  $r = 1, \dots, R$ , where  $R$  denotes the number of bootstrap replications. This provides the bootstrap distribution estimate as the distribution of  $\hat{\pi}^{*(r)} - \hat{\pi}$ ,  $r = 1, \dots, R$ , where, as before,  $\hat{\pi}$  denotes the unrestricted choice probability vector. We need to generate the bootstrap samples under the null, however. A naive way to achieve this would be to center it around the restricted estimator  $\hat{\eta}$ , that is

$$\hat{\pi}_{naive}^{*(r)} = \hat{\pi}^{*(r)} - \hat{\pi} + \hat{\eta}, \quad r = 1, \dots, R.$$

Recall that  $\hat{\eta}$  is the solution to

$$\begin{aligned} J_N &:= N \min_{\eta \in C} [\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta] \\ &= N \min_{\nu \in \mathbf{R}_+^H} [\hat{\pi} - A\nu]' \Omega [\hat{\pi} - A\nu]. \end{aligned}$$

But  $\hat{\pi}_{naive}^*$  is invalid due to standard results about the failure of the bootstrap in discontinuous models (e.g., Andrews (2000)). The “tightening” remedy is to center it instead around the tightened restricted estimator. More precisely, our procedure is as follows:

(i) Obtain the  $\tau_N$ -tightened restricted estimator  $\hat{\eta}_{\tau_n}$ , which solves

$$\begin{aligned} J_N &:= \min_{\eta \in C_{\tau_N}} N[\hat{\pi} - \eta]' \Omega [\hat{\pi} - \eta] \\ &= \min_{[\nu - \tau_N \mathbf{1}_H] \in \mathbf{R}_+^H} N[\hat{\pi} - A\nu]' \Omega [\hat{\pi} - A\nu] \end{aligned}$$

(ii) Calculate the bootstrap estimators under the restriction, using the recentering factor  $\hat{\eta}_{\tau_n}$  obtained in (i):

$$\hat{\pi}_{\tau_n}^{*(r)} := \hat{\pi}^{*(r)} - \hat{\pi} + \hat{\eta}_{\tau_n}, \quad r = 1, \dots, R.$$

(iii) Calculate the bootstrap test statistic by solving the following problem:

$$\begin{aligned} J_N^{*(r)}(\tau_N) &:= \min_{\eta \in C_{\tau_N}} N[\hat{\pi}_{\tau_n}^{*(r)} - \eta]' \Omega [\hat{\pi}_{\tau_n}^{*(r)} - \eta] \\ &= \min_{[\nu - \tau_N \mathbf{1}_H] \in \mathbf{R}_+^H} N[\hat{\pi}_{\tau_n}^{*(r)} - A\nu]' \Omega [\hat{\pi}_{\tau_n}^{*(r)} - A\nu], \end{aligned}$$

for  $r = 1, \dots, R$ .

(iv) Use the empirical distribution of  $J_N^{*(r)}(\tau_N), r = 1, \dots, R$  to obtain the critical value for  $J_N$ .

This method relies on a tuning parameter  $\tau_N$  which plays the role of a similar tuning parameter in the moment selection approach (namely, the parameter labeled  $\kappa_N$  in Andrews and Soares (2010)). In a simplified procedure in which the unrestricted choice probability estimate is obtained by simple sample frequencies, one reasonable choice would be

$$\tau_N = \sqrt{\frac{\log \underline{N}}{\underline{N}}}$$

where  $\underline{N}$  is the minimum of the ‘sample size’ across budgets:  $\underline{N} = \min_j N_j$  ( $N_j$  is the number of observations on Budget  $B_j$ : see (5.1)). The logarithmic penalization corresponds to the Bayes Information Criterion. The use of  $\underline{N}$  can probably be improved upon, but suffices to ensure validity of our inference.

## 7. TEST STATISTIC WITH SMOOTHING

In our empirical analysis in Section 8, the estimator  $\hat{\pi}$  is a standard kernel estimator, so the above formula needs to be modified. Let  $h_N(j)$  be the bandwidth applied to Budget  $B_j$ . Then an

appropriate choice of  $\tau_N$  is obtained by replacing  $N_j$  in the definition above with the “effective sample size”  $N_j h_N(j)$ . That is:

$$\tau_N = \sqrt{\frac{\log Nh}{Nh}}$$

where  $Nh = \min_j N_j h_N(j)$ . Strictly speaking, asymptotics with nonparametric smoothing involve bias, and the bootstrap does not solve the problem. A standard procedure is to claim that one used undersmoothing and can hence ignore the bias. We follow this convention.

To formally state the asymptotic theory behind our procedure with smoothing, let  $\mathbf{x}(j)$  denote the median (log) income levels for year  $j$ . We observe  $J$  random samples  $\left\{ \left( \{d_{i|j}\}_{i=1}^{I_j}, x_n(j) \right) \right\}_{n=1}^{N_j}, j = 1, \dots, J$ . Instead of sample frequency estimators, we use

$$\begin{aligned} \hat{\pi}_{i|j} &= \frac{\sum_{n=1}^{N_j} K\left(\frac{x_n(j) - \mathbf{x}(j)}{h_N(j)}\right) d_{i|j,n}}{\sum_{n=1}^{N_j} K\left(\frac{x_n(j) - \mathbf{x}(j)}{h_N(j)}\right)}, \\ \hat{\pi}_j &= (\hat{\pi}_{1|j}, \dots, \hat{\pi}_{I_j|j})', \\ \hat{\pi} &= (\hat{\pi}'_1, \dots, \hat{\pi}'_J)', \end{aligned}$$

(note the index  $j$  in  $d_{i|j,n}$  now refers to year), where  $x_n(j)$  is the log income of person  $n$ ,  $1 \leq n \leq N_j$  observed in year  $j$ . The kernel function  $K$  is assumed to be symmetric about zero, to integrate to 1, and to satisfy  $\int |K(z)| dz < \infty$  and  $\int z^2 K(z) dz < \infty$ . Needless to say,  $\hat{\pi}_{i|j}$  is a standard kernel regression estimator for

$$\pi_{i|j} := p_{i|j}(\mathbf{x}(j))$$

where

$$p_{i|j}(x) := \Pr(d_{i|j,n} = 1 | x_n(j) = x).$$

As before, we write  $\pi_j := (\pi_{1|j}, \dots, \pi_{I_j|j})'$  and  $\pi := (\pi'_1, \dots, \pi'_J)' = (\pi_{1|1}, \pi_{2|1}, \dots, \pi_{I_J|J})'$ . The smoothed version of  $J_N$  is computed using the above kernel estimator for  $\hat{\pi}$  in (3.1). We also replace the normalizing factor  $N$  with  $N \min_j h_N(j)$ , which is convenient for asymptotic analysis. Likewise,  $\tilde{J}_N(\tau_N)$  is obtained applying the same replacements to the formula (5.2), although generating  $\tilde{\eta}_{\tau_N}$  requires a slight modification. Let  $\hat{\eta}_{\tau_N}(j)$  be the  $j$ -th block of the vector  $\hat{\eta}_{\tau_N}$ , and  $\hat{S}_j$  be a consistent estimator for the asymptotic variance of  $\sqrt{N_j h_N(j)}(\hat{\pi}_j - \pi_j)$ . We use  $\tilde{\eta}_{\tau_N} = (\tilde{\eta}_{\tau_N}(1)', \dots, \tilde{\eta}_{\tau_N}(J)')$  for the smoothed version of  $\tilde{J}_N(\tau_N)$ , where  $\tilde{\eta}_{\tau_N}(j) := \hat{\eta}_{\tau_N}(j) + \frac{1}{\sqrt{N_j h_N(j)}} N(0, \hat{S}_j), j = 1, \dots, J$ .

Let  $\dot{h}$  ( $\ddot{h}$ ) denote the first (second) derivative of a function  $h$ . Recall that the modulus of continuity of  $h$  defined on  $\mathbf{R}$  at  $x_0$  is given by

$$\omega(h, t, x_0) := \sup |h(x) - h(x_0)| \quad s.t. \quad |x - x_0| \leq t$$

for  $t > 0$ .

**Condition 7.1.** Let  $\epsilon > 0$  and  $K < \infty$  be some constants, and  $\rho : [0, \infty] \rightarrow [0, \infty]$  some function such that  $\rho(t) \downarrow 0$  as  $t \downarrow 0$ . The following holds:

- (i)  $\pi \in C$ ;
- (ii)  $\text{var}(b_k(j)'d_{j,n}|x_n(j) = \mathbf{x}(j)) \geq \epsilon$  holds for all  $1 \leq k \leq m$  for some  $j = 1, \dots, J$ ;
- (iii)  $x_n(j)$  is continuously distributed with density  $f_j$ , and  $f_j(\mathbf{x}(j)) \geq \epsilon$  for every  $1 \leq j \leq J$ ;
- (iv)  $p_{i|j}$  is twice differentiable and  $\omega(\ddot{p}_{i|j}, t, \mathbf{x}(j)) \leq \rho(t), t \geq 0, \sup |\dot{p}_{i|j}| \leq K$  and  $\sup |p_{i|j}| \leq K$  for every  $1 \leq i \leq I_j$  and  $1 \leq j \leq J$ ;
- (v)  $f_j$  is twice differentiable and  $\omega(\ddot{f}_j, t, \mathbf{x}(j)) \leq \rho(t), t \geq 0, \sup f_j \leq K, \sup |\dot{f}_j| \leq K$  and  $\sup |\ddot{f}_j| \leq K$  for every  $1 \leq j \leq J$ .

In what follows,  $F_j$  signifies the joint distribution of  $(d_{i|j,n}, x_n(j))$ . Let  $\mathcal{F}$  be the set of all  $(F_1, \dots, F_J)$  that satisfy Condition 7.1 for some  $(\epsilon, K, \rho(\cdot))$ .

**Corollary 7.1.** Choose  $\tau_N$  and  $h_N(j), j = 1, \dots, J$  so that  $\tau_N \downarrow 0, h_N(j) = o(N^{-\frac{1}{5}})$  and  $\sqrt{N \min_j h_N(j)} \tau_N \uparrow \infty$ . Also let  $\Omega$  be diagonal where all the diagonal elements are positive. Then under Assumption 5.1

$$\liminf_{N \rightarrow \infty} \inf_{(F_1, \dots, F_J) \in \mathcal{F}} \Pr\{J_N \leq \hat{c}_{1-\alpha}\} = 1 - \alpha$$

where  $\hat{c}_{1-\alpha}$  is the  $1 - \alpha$  quantile of  $\tilde{J}_N(\tau_N), 0 \leq \alpha \leq \frac{1}{2}$ .

The proof of Corollary 7.1 resembles the proof of Theorem 5.1, except that the asymptotic normality proof in Bierens (1987, Section 2.2) and Condition 7.1 are used to obtain triangular array CLT's and uniform consistency results.

## 8. EMPIRICAL APPLICATION

We apply our methods to data from the U.K. Family Expenditure Survey, the same data used by Blundell, Browning, and Crawford (2008, BBC henceforth). To facilitate comparison of results, we use the same subset of these data as they do, namely the time periods from 1975 through 1999, households with a car and at least one child, and also the same composite goods, namely food, nondurable consumption goods, and services (with total expenditure on the three standing in for income).

For each year, we extract the budget corresponding to that year's median expenditure. We estimate the distribution of demand on that budget by kernel density estimation applied to budget shares, where the kernel is normal, the bandwidth is chosen according to Silverman's rule of thumb,

and the underlying distance metric is  $\log(\text{income})$ . Like BBC, we assume that all consumers in one year face the same prices, and we use their price data. While budgets have a tendency to move outward over time, we find that there is substantial overlap of budgets. Thus, our analysis is not subject to the frequently reported problem that revealed preference tests are near vacuous because income gains dominate relative price changes. At the same time, the tendency of budgets to move outward means that budgets which are more than a few years apart rarely overlap, making the refinement of our crawling algorithm via Proposition 4.1 very powerful in this application.

It is computationally prohibitive to test stochastic rationality on 25 periods at once. We work with all possible sets of eight consecutive periods, a problem size that can be very comfortably computed. This leads to a collection of 18 matrices  $(X, A)$ . Testing problems were of vastly different complexity, with the size of the matrix  $A$  ranging from  $(14 \times 21)$  to  $(67 \times 149570)$  and  $(64 \times 177352)$ ; thus, there were up to 67 patches and up to 177352 rational choice types. Over this range of problem sizes, time required to compute  $A$  varied from negligible to several hours with the crawling algorithm. Time required to compute the larger matrices improves by a factor of about 100 using Proposition 4.1. Time required to compute  $J_N$  varied from negligible to about one minute, with problem size  $(67 \times 149570)$  being the hardest. Thus, our informal assessment is that for computation of  $J_N$ , increasing  $I$  (and therefore the dimensionality of the quadratic programming problem's objective function) is more costly than increasing  $H$  (and therefore the number of linear constraints).

Figures 1 and 2 provide a visual illustration using the 1975-82 periods. Figure 1 visualizes the relevant budget sets; Figure 2 illustrates patches on the 1982 budget set. There is substantial overlap across budgets, with the 1975-1982 periods generating 50 distinct "patches" and a total of 42625 choice types. The reason is that due to the 1970's recession, budgets initially contract and then recover over this time period, generating an intricate pattern of mutual overlaps.

For each testing problem, we computed  $X$ ,  $A$ ,  $J_N$ , as well as critical values and p-values using the tightened bootstrap algorithm with  $R = 2000$  resamples. Results are summarized in Table 1. We find that  $J_N = 0$  in only one case; that is, only in one case are the estimated choice probabilities rationalizable. However, violations of stochastic rationality are by and large not significant. We get one p-value below 10%, illustrating that in principle our test has some power, though this p-value must of course be seen in the context of the multiple hypothesis tests reported in table 1.<sup>12</sup> Also, the p-values seem to exhibit strong serial dependence. This is as expected – recall that any two consecutive

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<sup>12</sup>An increase of the bootstrap size to 10000 (not reported) confirms a p-value of under 10% for this test statistic seen in isolation. Of course, a single rejection at the 10% level is well within expectations for joint testing of 18 hypotheses.



periods	I	H	N	$J_N$	p-value	10% c.v.	5% c.v.
<b>75-82</b>	50	42625	1229	.23	96%	3.1	4.0
<b>76-83</b>	67	149570	1209	.25	95%	2.5	3.3
<b>77-84</b>	55	97378	1160	.22	94%	2.0	2.5
<b>78-85</b>	49	61460	1108	.22	93%	1.7	2.3
<b>79-86</b>	43	38521	1033	.24	69%	.76	.94
<b>80-87</b>	35	7067	1033	.16	77%	2.1	3.4
<b>81-88</b>	27	615	995	.16	62%	.67	.82
<b>82-89</b>	14	21	987	0	100%	.24	.34
<b>83-90</b>	14	21	869	.00015	50%	2.1	3.5
<b>84-91</b>	16	63	852	.15	71%	2.6	3.8
<b>85-92</b>	27	1131	852	1.75	22%	3.1	4.3
<b>86-93</b>	43	10088	828	2.15	17%	2.9	3.8
<b>87-94</b>	49	55965	828	2.59	18%	3.5	4.6
<b>88-95</b>	52	75318	828	2.23	15%	2.8	3.6
<b>89-96</b>	64	177352	795	1.61	23%	2.7	3.6
<b>90-97</b>	46	22365	715	1.74	9.1%	1.69	2.2
<b>91-98</b>	39	9400	715	2.17	16%	2.8	3.5
<b>92-99</b>	21	225	715	2.58	10.2%	2.6	3.4

TABLE 1. Numerical Results of Testing Rationality Using FES data.

p-values were computed from two subsets of data that overlap in seven of eight periods. Our overall conclusion is that observed choice probabilities differ from rationalizable ones but not significantly so.

## 9. FURTHER APPLICATIONS AND EXTENSIONS

**9.1. Partial Identification of  $\nu$ .** Our setting gives rise to an identified set  $\mathbf{H}_\nu$  for weights  $\nu$  over rational choice types

$$\mathbf{H}_\nu = \{\nu \in \Delta^{H-1} : A\nu = \pi\},$$

and our main test could be interpreted as a specification test for the null hypothesis  $H_0 : \mathbf{H}_\nu \neq \emptyset$ . Estimation of  $\mathbf{H}_\nu$  is, therefore, a natural issue. We focus on it somewhat less because the identified set is a collection of distributions over lists of choice behaviors and, at least in our application, is not

FIGURE 1. Visualization of 1975-1982 budgets

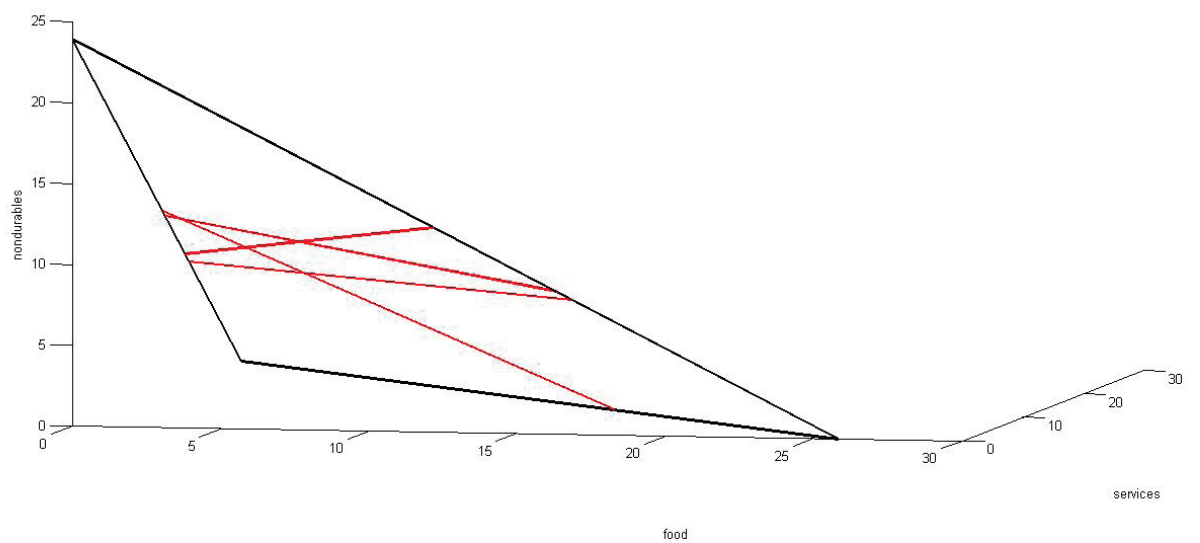
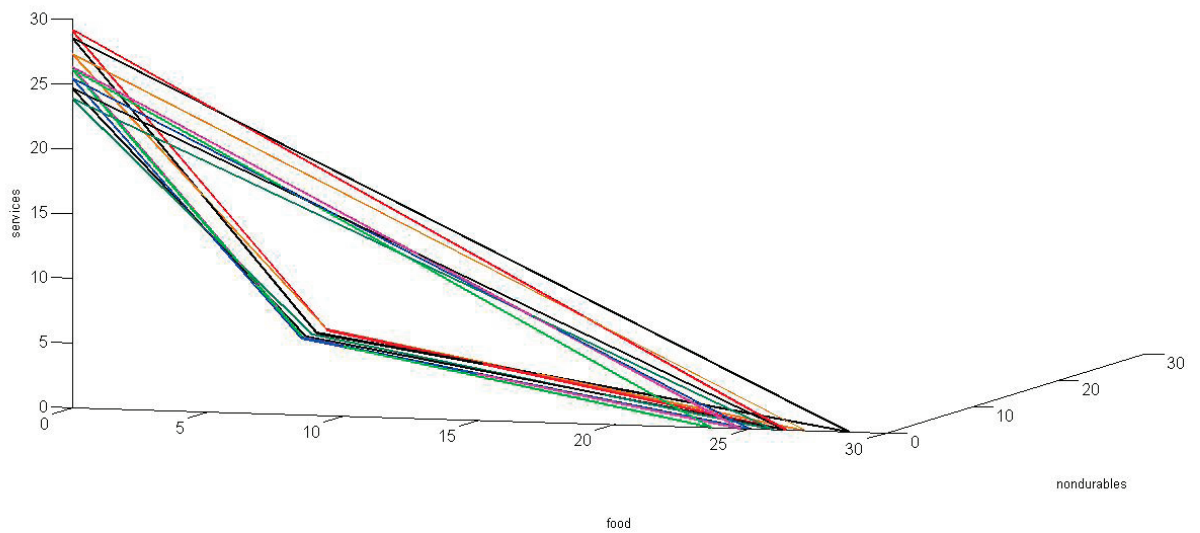


FIGURE 2. Visualization of "patches" on the 1982 budget

immediately interpretable in terms of structural parameters of interest. This caveat might not apply to other applications, however.

If our model is misspecified, then  $\mathbf{H}_\nu$  is empty. To generate an estimand that is nonempty by construction, define

$$\mathbf{H}_\nu(\Omega) = \arg \min_{\nu \in \Delta^{H-1}} \{(A\nu - \pi)' \Omega (A\nu - \pi)\},$$

This set is independent of the choice of the weighting matrix  $\Omega$  (as long as  $\Omega$  is positive definite), and then coincides with  $\mathbf{H}_\nu$  as previously defined, iff the latter is nonempty. But it also defines a coherent notion of “pseudo-true (partially identified) distribution of rational types” if the data are not rationalizable.

$\mathbf{H}_\nu$  is a singleton iff  $\nu$  is point identified, which will rarely be the case in interesting applications of our framework. Indeed, in our application,  $\mathbf{H}_\nu(\Omega)$  is a high-dimensional, convex polyhedron even in cases where  $\pi$  is not rationalizable. That is, unlike in many other applications of moment inequalities, failure of the sample criterion function to attain a minimal value of zero does not make the sample analog of the identified set a singleton.

Explicit computation of  $\mathbf{H}_\nu(\Omega)$  is demanding. Our suggestion is to write

$$\mathbf{H}_\nu(\Omega) = \left\{ \nu \in \Delta^{H-1} : (A\nu - \pi)' \Omega (A\nu - \pi) - \min_{v \in \Delta^{H-1}} \{(Av - \pi)' \Omega (Av - \pi)\} = 0 \right\}$$

and compute a plug-in estimator that replaces  $\pi$  with  $\hat{\pi}$ . Noting that we showed how to compute the inner minimum, computation of these estimators could utilize methods based on support vector machines as developed in current work by Bar and Molinari (2012). An appropriately modified version of the method described in Sections 5 and 6 can be applied to inference on  $\mathbf{H}_\nu$  or elements of  $\mathbf{H}_\nu$ , though we leave its detailed analysis for future research.

**9.2. Partial Identification of Counterfactual Choices.** The toolkit developed in this paper is also useful for counterfactual analysis. At the most general level, to bound the value of any function  $f(\nu)$  subject to the constraint that  $\nu$  rationalizes the observed data, solve the program

$$\min_{\nu \in \mathbf{R}_+^H} / \max_{\nu \in \mathbf{R}_+^H} f(\nu) \quad \text{s.t. } A\nu = \hat{\eta},$$

recalling that  $\hat{\eta} = \hat{\pi}$  whenever  $\hat{\pi}$  is rationalizable.<sup>13</sup> Some interesting applications emerge by restricting attention to linear functions  $f(\nu) = e'\nu$ , in which case the bounds are furthermore relatively easy to compute because the program is linear. We briefly discuss bounding demand under a counterfactual

<sup>13</sup>Equivalently, one could use the previous subsection’s notation to write

$$\min / \max f(v) \quad \text{s.t. } v \in \mathbf{H}_\nu(\Omega),$$

which is more similar to the way that similar problems are stated by Manski (e.g., Manski (2007)).

budget, e.g. in order to measure the impact of policy intervention. This is close in spirit to bounds reported by Blundell et al. (2008; see also Cherchye et al. (2009) and references therein) as well as Manski (2007, 2013).

In this subsection only, assume that demand on budget  $B_J$  is not observed but is to be bounded. Write

$$A = \begin{bmatrix} A_{-J} \\ A_J \end{bmatrix}, \pi = \begin{bmatrix} \pi_{-J} \\ \pi_J \end{bmatrix}, \pi_J = \begin{bmatrix} \pi_{1|J} \\ \vdots \\ \pi_{I_J|J} \end{bmatrix}$$

and let  $e_i$  signify the  $i$ -th unit vector. We will begin by bounding components of the vector  $\pi_J$ .

**Corollary 9.1.**  $\pi_{i|J}$  is bounded by

$$\underline{\pi}_{i|J} \leq \pi_{i|J} \leq \bar{\pi}_{i|J}$$

where

$$\begin{aligned} \underline{\pi}_{i|J} &= \min \{ e_i' A_J \nu \} \quad \text{s.t.} \quad A_{-J} \nu = \pi_{-J}, \quad \nu \geq 0 \\ \bar{\pi}_{i|J} &= \max \{ e_i' A_J \nu \} \quad \text{s.t.} \quad A_{-J} \nu = \pi_{-J}, \quad \nu \geq 0. \end{aligned}$$

In this paper's application, the patches  $x_{i|J}$ , and hence the probabilities  $\pi_{i|J}$ , are not of intrinsic interest. However, they might be the object of interest in applications where the choice problem was discrete to begin with. Indeed, the above is the bounding problem further analyzed in Sher et al. (2011).

Next, let  $\delta(J) = \mathbb{E}[\arg \max_{y \in B_J} u(y)]$ , thus the vector  $\delta(J)$  with typical component  $\delta_k(J)$  denotes expected demand in choice problem  $B_J$ . Define the vectors

$$\begin{aligned} \underline{d}_k(J) &: = [\underline{d}_k(1|J), \dots, \underline{d}_k(I_J|J)] \\ \bar{d}_k(J) &: = [\bar{d}_k(1|J), \dots, \bar{d}_k(I_J|J)] \end{aligned}$$

with components

$$\begin{aligned} \underline{d}_k(i|J) &: = \min \{ y_k : y \in x_{i|J} \}, \quad 1 \leq i \leq I_J \\ \bar{d}_k(i|J) &: = \max \{ y_k : y \in x_{i|J} \}, \quad 1 \leq i \leq I_J, \end{aligned}$$

thus these vectors list minimal respectively maximal consumption of good  $k$  on the different patches within  $B_J$ . Computing  $(\underline{d}_k(i|J), \bar{d}_k(i|J))$  is a linear programming exercise. Then we have:

**Corollary 9.2.** *Expected demand for good  $k$  on budget  $B_J$  is bounded by*

$$\underline{\delta}_k(J) \leq \delta_k(J) \leq \bar{\delta}_k(J),$$

where

$$\begin{aligned} \underline{\delta}_k(J) & : = \min \underline{d}_k(J) A_J \nu \quad \text{s.t.} \quad A_{-J} \nu = \pi_J, \quad \nu \geq 0 \\ \bar{\delta}_k(J) & : = \max \bar{d}_k(J) A_J \nu \quad \text{s.t.} \quad A_{-J} \nu = \pi_J, \quad \nu \geq 0. \end{aligned}$$

Finally, consider bounding the c.d.f.  $F_k(z) = \Pr(y_k \leq z)$ . This quantity must be bounded in two steps. The event  $(y_k \leq z)$  will in general not correspond to a precise set of patches, that is, it is not measurable with respect to (the algebra generated by)  $\{x_{1|J}, \dots, x_{I_J|J}\}$ . An upper bound on  $F_k(z)$  will derive from an upper bound on the joint probability of all patches  $x_{i|J}$  s.t.  $y_k \leq z$  holds for *some*  $y \in x_{i|J}$ . Similarly, a lower bound will derive from bounding the joint probability of all patches  $x_{i|J}$  s.t.  $y_k \leq z$  holds for *all*  $y \in x_{i|J}$ .<sup>14</sup> We thus have:

**Corollary 9.3.** *For  $k = 1, \dots, K$  and  $z \geq 0$ ,  $F_k(z)$  is bounded from below by*

$$\begin{aligned} \min_{\nu \in \mathbf{R}_+^H} \sum_{i \in \{1, \dots, I_J\}: \underline{d}_k(i|J) \leq z} e'_i A_J \nu \\ \text{s.t.} \quad A_{-J} \nu = \pi_{-J} \end{aligned}$$

and from above by

$$\begin{aligned} \max_{\nu \in \mathbf{R}_+^H} \sum_{i \in \{1, \dots, I_J\}: \bar{d}_k(i|J) \leq z} e'_i A_J \nu \\ \text{s.t.} \quad A_{-J} \nu = \pi_{-J}, \end{aligned}$$

where  $(\underline{d}_k(i|J), \bar{d}_k(i|J))$  are defined as before.

While both the lower and the upper bound, seen as functions of  $z$ , will themselves be proper c.d.f.'s, they are not in general feasible distributions of demand for  $y_k$ . That is, the bounds are sharp pointwise but not uniformly. Also, bounds on a wide range of parameters such as the variance of demand follow from the above bounds on the c.d.f. through results in Stoye (2010). However, because the bounds on the c.d.f. are not uniform, these derived bounds will be valid but not necessarily sharp.

When trained on this paper's empirical application, these bounds are uncomfortably wide, motivating the search for nonparametric refinements that lead to narrower bounds without tightly

<sup>14</sup>These definitions correspond to inner and outer measure, as well as to hitting and containment probability.

constraining heterogeneity. This search, as well as the development of inference procedures for the bounds, are the subject of ongoing research. We also note that in his recent analysis of optimal taxation of labor, Manski (2013) uses our computational tools to find informative bounds.

**9.3. Choice from Binary Sets.** The methods developed in this paper, including the two extensions just discussed, immediately apply to nonparametric analysis of random discrete choice. Indeed, the initial discretization step that characterizes our analysis of a demand system is superfluous in this case. We briefly elaborate on one salient application that has received attention in the literature, namely the case where choice probabilities for pairs of options,

$$\pi_{ab} := \Pr(a \text{ is chosen from } \{a, b\})$$

are observed for all pairs of choice objects  $\{a, b\}$  drawn from some finite, universal set  $\mathcal{A}$ .

Finding abstract conditions under which a set of choice probabilities  $\{\pi_{ab} : a, b \in \mathcal{A}\}$  is rationalizable has been the objective of two large, disjoint literatures, one in economics and one in operations research. See Fishburn (1992) for a survey of these literatures and Manski (2007) for a recent discussion of the substantive problem. There exists a plethora of necessary conditions, most famously Marschak’s (1960) triangle condition, which can be written as

$$\pi_{ab} + \pi_{bc} + \pi_{ca} \leq 2, \forall a, b, c \in \mathcal{A}.$$

This condition is also sufficient for rationalizability if  $\mathcal{A}$  contains at most 5 elements (Dridi (1980)). Conditions that are both necessary and sufficient in general have proved elusive. We do not discover abstract such conditions either, but provide the toolkit to numerically resolve the question in complicated cases, including a statistical test that applies whenever probabilities are estimated rather than perfectly observed. To see this, define  $J = (\#\mathcal{A})(\#\mathcal{A} - 1)/2$  “budgets” that correspond to distinct pairs  $a, b \in \mathcal{A}$ , and let the vector  $X$  (of length  $I = 2J$ ) stack these budgets, where the ordering of budgets is arbitrary and options within a budget are ordered according to a preassigned ordering on  $\mathcal{A}$ . Each rational type (and thus, column of the matrix  $A$ ) then corresponds to an ordering of the elements of  $\mathcal{A}$  and can be characterized by a binary  $I$ -vector with just the same interpretation as before. As before, an  $I$ -vector of choice probabilities  $\pi$  whose components correspond to components of  $X$  is rationalizable iff  $A\nu = \pi$  for some  $\nu \in \Delta^{H-1}$ . All methods developed in this paper apply immediately.

To illustrate, let  $\mathcal{A} = \{a, b, c\}$ , then one can write

$$X = \begin{bmatrix} a \\ b \\ b \\ c \\ c \\ a \end{bmatrix}, \pi = \begin{bmatrix} \pi_{ab} \\ \pi_{ba} \\ \pi_{bc} \\ \pi_{cb} \\ \pi_{ca} \\ \pi_{ac} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix},$$

and it is readily verified that  $A\nu = \pi$  for some  $\nu \in \Delta^5$  iff both  $\pi_{ab} + \pi_{bc} + \pi_{ca} \leq 2$  and  $\pi_{cb} + \pi_{ba} + \pi_{ac} \leq 2$ , confirming sufficiency of the triangle condition. More generally, the matrix  $A$  has  $H = (\#\mathcal{A})!$  columns. This limits computational feasibility of our approach, but note that the set of orderings of elements of  $\mathcal{A}$  is easily characterized, so that computation time per column of  $A$  will be low.

## 10. CONCLUSION

This paper presented asymptotic theory and computational tools for nonparametric testing of Random Utility Models. Again, the null to be tested was that data were generated by a RUM, interpreted as describing a heterogeneous population, where the only restrictions imposed on individuals' behavior were "more is better" and SARP. In particular, we allowed for unrestricted, unobserved heterogeneity and stopped far short of assumptions that would recover invertibility of demand. As a result, the distribution over utility functions in the population is left (very) underidentified. We showed that testing the model is nonetheless possible. The method is easily adapted to choice problems that are discrete to begin with, and one can easily impose more, or also fewer, restrictions at the individual level.

Possibilities for extensions and refinements abound. We close by mentioning some salient issues.

- (1) The methods discussed in this section are computationally intensive. The proposed algorithms work for a reasonably sized problem, though it is important to make further improvements in the algorithms if one wishes to deal with a problem that is large, say in terms of the number of budgets.
- (2) We restricted attention to finite sets of budgets. The extension to infinitely many budgets would be of obvious interest. Theoretically, it can be handled by considering an appropriate discretization argument (McFadden (2005)). For the proposed projection-based econometric methodology, such an extension requires evaluating choice probabilities locally over points in the space of  $p$  via nonparametric smoothing, then use the choice probability estimators in the calculation of the  $J_N$  statistic. The

asymptotic theory then needs to be modified. Another approach that can mitigate the computational constraint is to consider a partition of the space of  $p$  such that  $\mathbf{R}_+^K = \mathcal{P}_1 \cup \mathcal{P}_2 \cdots \cup \mathcal{P}_M$ . Suppose we calculate the  $J_N$  statistic for each of these partitions. Given the resulting  $M$  statistics, say  $J_N^1, \dots, J_N^M$ , we can consider  $J_N^{\max} := \max_{1 \leq m \leq M} J_N^m$  or a weighted average of them. These extensions and their formal statistical analysis are of practical interest.

(3) It might frequently be desirable to control for observable covariates to guarantee the homogeneity of the distribution of unobserved heterogeneity. Once again, this requires incorporating nonparametric smoothing in estimating choice probabilities, then averaging the corresponding  $J_N$  statistics over the covariates. This extension will be pursued.

(4) The econometric techniques outlined here can be potentially useful in much broader contexts. Again, our proposed hypothesis test can be regarded as specification test for a moment inequalities model. The proposed statistic  $J_N$  is an inequality analogue of goodness-of-fit statistics such as Hansen's (1982) overidentifying restrictions test statistic. Existing proposals for specification testing in moment inequality models (Andrews and Guggenberger (2009), Andrews and Soares (2010), Bugni, Canay, and Shi (2013), Romano and Shaikh (2008)) use a similar test statistic but work with  $\mathcal{H}$ -representations. In settings in which theoretical restrictions inform a  $\mathcal{V}$ -representation of a cone, the  $\mathcal{H}$ -representation will typically not be available in practice. We expect that our method can be used in many such cases.

## 11. APPENDIX A: PROOFS

**Proof of Proposition 4.1.** We begin with some preliminary observations. Throughout this proof,  $c(B_i)$  denotes the object actually chosen from budget  $B_i$ .

(i) If there is a choice cycle of any finite length, then there is a cycle of length 2 or 3 (where a cycle of length 2 is a WARP violation). To see this, assume there exists a length  $N$  choice cycle  $c(B_i) \succ c(B_j) \succ c(B_k) \succ \dots \succ c(B_i)$ . If  $c(B_k) \succ c(B_i)$ , then a length 3 cycle has been discovered. Else, there exists a length  $N - 1$  choice cycle  $c(B_i) \succ c(B_k) \succ \dots \succ c(B_i)$ . The argument can be iterated until  $N = 4$ .

(ii) Call a length 3 choice cycle *irreducible* if it does not contain a length 2 cycle. Then a choice pattern is rationalizable iff it contains no length 2 cycles and also no irreducible length 3 cycles. (In particular, one can ignore reducible length 3 cycles.) This follows trivially from (i).



(iii) Let  $J = 3$  and  $M = 1$ , i.e. assume there are three budgets but two of them fail to intersect. Then any length 3 cycle is reducible. To see this, assume w.l.o.g. that  $B_1$  is below  $B_3$ , thus  $c(B_3) \succ c(B_1)$  by monotonicity. If there is a choice cycle, we must have  $c(B_1) \succ c(B_2) \succ c(B_3)$ .  $c(B_1) \succ c(B_2)$  implies that  $c(B_2)$  is below  $B_1$ , thus it is below  $B_3$ .  $c(B_2) \succ c(B_3)$  implies that  $c(B_3)$  is below  $B_2$ . Thus, choice from  $(B_2, B_3)$  violates WARP.

We are now ready to prove the main result. The nontrivial direction is “only if,” thus it suffices to show the following: If choice from  $(B_1, \dots, B_{J-1})$  is rationalizable but choice from  $(B_1, \dots, B_J)$  is not, then choice from  $(B_{M+1}, \dots, B_J)$  cannot be rationalizable. By observation (ii), if  $(B_1, \dots, B_J)$  is not rationalizable, it contains either a 2-cycle or an irreducible 3-cycle. Because choice from all triplets within  $(B_1, \dots, B_{J-1})$  is rationalizable by assumption, it is either the case that some  $(B_i, B_J)$  constitutes a 2-cycle or that some triplet  $(B_i, B_k, B_J)$ , where  $i < k$  w.l.o.g., reveals an irreducible choice cycle. In the former case,  $B_i$  must intersect  $B_J$ , hence  $i > M$ , hence the conclusion. In the latter case, if  $k \leq M$ , the choice cycle must be a 2-cycle in  $(B_i, B_k)$ , contradicting rationalizability of  $(B_1, \dots, B_{J-1})$ . If  $i \leq M$ , the choice cycle is reducible by (iii). Thus,  $i > M$ , hence the conclusion.  $\square$

**Proof of Lemma 5.1.** By the Minkowski-Weyl theorem

$$C = \{t \in \mathbf{R}^I : Bt \leq 0\}.$$

Letting  $\nu_\tau = \nu - \tau \mathbf{1}_H$ , we have

$$\begin{aligned} C_\tau &= \{A\nu | \nu_\tau \geq 0\} \\ &= \{A[\nu_\tau + \tau \mathbf{1}_H] | \nu_\tau \geq 0\} \\ &= C \oplus \tau A \mathbf{1}_H \\ &= \{t : t - \tau A \mathbf{1}_H \in C\} \end{aligned}$$

where  $\oplus$  signifies Minkowski sum. Define

$$\phi = -BA \mathbf{1}_H.$$

Using the  $\mathcal{H}$ -representation of  $C$ ,

$$\begin{aligned} C_\tau &= \{t : B(t - \tau A \mathbf{1}_H) \leq 0\} \\ &= \{t : Bt \leq -\tau \phi\}. \end{aligned}$$

Note that the above definition of  $\phi$  implies  $\phi \in \text{col}(B)$ . Also define

$$\begin{aligned}\Phi &:= -BA \\ &= - \begin{bmatrix} b'_1 \\ \vdots \\ b'_m \end{bmatrix} [a_1, \dots, a_H] \\ &= \{\Phi_{kh}\}\end{aligned}$$

where  $\Phi_{kh} = b'_k a_h$ ,  $1 \leq k \leq m$ ,  $1 \leq h \leq H$  and let  $e_h$  be the  $h$ -th standard unit vector in  $\mathbf{R}^H$ . Since  $e_h \geq 0$ , the  $\mathcal{V}$ -representation of  $C$  implies that  $Ae_h \in C$ , and thus

$$BAe_h \leq 0$$

by its  $\mathcal{H}$ -representation. Therefore

$$(11.1) \quad \Phi_{kh} = -e'_k B A e_h \geq 0, \quad 1 \leq k \leq m, 1 \leq h \leq H.$$

Write  $\phi = (\phi_1, \dots, \phi_m)'$ . We now show that  $\phi_k \neq 0$  for all  $1 \leq k \leq m$ . To see it, note

$$\begin{aligned}\phi &= \Phi \mathbf{1}_H \\ &= \begin{bmatrix} \sum_{h=1}^H \Phi_{1h} \\ \vdots \\ \sum_{h=1}^H \Phi_{mh} \end{bmatrix}\end{aligned}$$

But unless we have the case of  $\text{rank}(B) = 1$  (in which case the proof is trivial) it cannot be that

$$a_j \in \{x : b'_k x = 0\} \quad \text{for all } j$$

Therefore for each  $k$ ,  $\Phi_{kh} = b'_k a_h$  is nonzero at least for one  $h$ ,  $1 \leq h \leq H$ . Since (11.1) implies that all of  $\{\Phi_{kh}\}_{h=1}^H$  are non-negative, we conclude that

$$\phi_k = \sum_{h=1}^H \Phi_{kh} > 0$$

for all  $k$ . We now have

$$C_\tau = \{t : Bt \leq -\tau\phi\}, \quad \phi > 0$$

where the strict vector inequality is meant to hold element-by-element. □

**Proof of Theorem 5.1.** By applying the Minkowski-Weyl theorem and Lemma 5.1 to  $J_N$  and  $\tilde{J}_N(\tau_N)$ , we see that our procedure is equivalent to comparing

$$J_N = \min_{t \in \mathbf{R}^I: Bt \leq 0} N[\hat{\pi} - t]' \Omega [\hat{\pi} - t]$$

to the  $1 - \alpha$  quantile of the distribution of

$$\tilde{J}_N(\tau_N) = \min_{t \in \mathbf{R}^I: Bt \leq -\tau_N \phi} N[\tilde{\eta}_{\tau_N} - t]' \Omega [\tilde{\eta}_{\tau_N} - t]$$

with  $\phi \in \mathbf{R}_{++}^m$ , where

$$\begin{aligned} \tilde{\eta}_{\tau_N} &= \hat{\eta}_{\tau_N} + \frac{1}{\sqrt{N}} N(0, \hat{S}), \\ \hat{\eta}_{\tau_N} &= \operatorname{argmin}_{t \in \mathbf{R}^I: Bt \leq -\tau_N \phi} N[\hat{\pi} - t]' \Omega [\hat{\pi} - t]. \end{aligned}$$

Suppose  $B$  has  $m$  rows and  $\operatorname{rank}(B) = \ell$ . Define an  $\ell \times m$  matrix  $K$  such that  $KB$  is a matrix whose rows consist of a basis of the row space  $\operatorname{row}(B)$ . Also let  $M$  be an  $(I - \ell) \times I$  matrix whose rows form an orthonormal basis of  $\ker B = \ker(KB)$ , and define  $P = \begin{pmatrix} KB \\ M \end{pmatrix}$ . Finally, let  $\hat{g} = B\hat{\pi}$  and  $\hat{h} = M\hat{\pi}$ . Then

$$\begin{aligned} J_N &= \min_{Bt \leq 0} N \left[ \begin{pmatrix} KB \\ M \end{pmatrix} (\hat{\pi} - t) \right]' P^{-1'} \Omega P^{-1} \left[ \begin{pmatrix} KB \\ M \end{pmatrix} (\hat{\pi} - t) \right] \\ &= \min_{Bt \leq 0} N \begin{pmatrix} K[\hat{g} - Bt] \\ \hat{h} - Mt \end{pmatrix}' P^{-1'} \Omega P^{-1} \begin{pmatrix} K[\hat{g} - Bt] \\ \hat{h} - Mt \end{pmatrix}. \end{aligned}$$

Let

$$\mathcal{U}_1 = \left\{ \begin{pmatrix} K\gamma \\ h \end{pmatrix} : \gamma = Bt, h = Mt, Bt \leq 0, t \in \mathbf{R}^I \right\}$$

then writing  $\alpha = KBt$  and  $h = Mt$ ,

$$J_N = \min_{\begin{pmatrix} \alpha \\ h \end{pmatrix} \in \mathcal{U}_1} N \begin{pmatrix} K\hat{g} - \alpha \\ \hat{h} - h \end{pmatrix}' P^{-1'} \Omega P^{-1} \begin{pmatrix} K\hat{g} - \alpha \\ \hat{h} - h \end{pmatrix}.$$

Also define

$$\mathcal{U}_2 = \left\{ \begin{pmatrix} K\gamma \\ h \end{pmatrix} : \gamma \leq 0, \gamma \in \operatorname{col}(B), h \in \mathbf{R}^{I-\ell} \right\}$$

where  $\operatorname{col}(B)$  denotes the column space of  $B$ . Obviously  $\mathcal{U}_1 \subset \mathcal{U}_2$ . Moreover,  $\mathcal{U}_2 \subset \mathcal{U}_1$  holds. To see this, let  $\begin{pmatrix} K\gamma^* \\ h^* \end{pmatrix}$  be an arbitrary element of  $\mathcal{U}_2$ . We can always find  $t^* \in \mathbf{R}^I$  such that  $K\gamma^* = KBt^*$ .

Define

$$t^{**} := t^* + M'h^* - M'Mt^*$$

then  $Bt^{**} = Bt^* = \gamma^* \leq 0$  and  $Mt^{**} = Mt^* + MM'h^* - MM'Mt^* = h^*$ , therefore  $\begin{pmatrix} K\gamma^* \\ h^* \end{pmatrix}$  is an element of  $\mathcal{U}_1$  as well. Consequently,

$$\mathcal{U}_1 = \mathcal{U}_2.$$

We now have

$$\begin{aligned} J_N &= \min_{\begin{pmatrix} \alpha \\ h \end{pmatrix} \in \mathcal{U}_2} N \begin{pmatrix} K\hat{g} - \alpha \\ \hat{h} - h \end{pmatrix}' P^{-1}' \Omega P^{-1} \begin{pmatrix} K\hat{g} - \alpha \\ \hat{h} - h \end{pmatrix} \\ &= N \min_{\begin{pmatrix} \alpha \\ y \end{pmatrix} \in \mathcal{U}_2} \begin{pmatrix} K\hat{g} - \alpha \\ y \end{pmatrix}' P^{-1}' \Omega P^{-1} \begin{pmatrix} K\hat{g} - \alpha \\ y \end{pmatrix}. \end{aligned}$$

Define

$$T(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}' P^{-1}' \Omega P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathbf{R}^\ell, y \in \mathbf{R}^{I-\ell},$$

and

$$t(x) := \min_{y \in \mathbf{R}^{I-\ell}} T(x, y), \quad s(g) := \min_{\gamma \leq 0, \gamma \in \text{col}(B)} t(K[g - \gamma]).$$

It is easy to see that  $t : \mathbf{R}^\ell \rightarrow \mathbf{R}_+$  is a positive definite quadratic form. We can write

$$\begin{aligned} J_N &= N \min_{\gamma \leq 0, \gamma \in \text{col}(B)} t(K[\hat{g} - \gamma]) \\ &= N s(\hat{g}) \\ &= s(\sqrt{N}\hat{g}). \end{aligned}$$

We now show that tightening can turn non-binding inequality constraints into binding ones but not vice versa. Note that, as will be seen below, this observation uses diagonality of  $\Omega$  and the specific geometry of the cone  $C$ . Let  $\hat{\gamma}_{\tau_N}^k$ ,  $\hat{g}^k$  and  $\phi^k$  denote the  $k$ -th elements of  $\hat{\gamma}_{\tau_N} = B\hat{\eta}_{\tau_N}$ ,  $\hat{g}$  and  $\phi$ . Moreover, define  $\gamma_\tau(g) = [\gamma^1(g), \dots, \gamma^m(g)]' = \text{argmin}_{\gamma \leq -\tau\phi, \gamma \in \text{col}(B)} t(K[g - \gamma])$  for  $g \in \text{col}(B)$ . Then  $\hat{\gamma}_{\tau_N} = \gamma_{\tau_N}(\hat{g})$ . Finally, define  $\beta_\tau(g) = \gamma_\tau(g) + \tau\phi$  for  $\tau > 0$  and let  $\beta_\tau^k(g)$  denote its  $k$ -th element. Now we show that for each  $k$  and for some  $\delta > 0$ ,

$$\beta_\tau^k(g) = 0$$

if  $|g^k| \leq \tau\delta$  and  $g^j \leq \tau\delta, 1 \leq j \leq m$ . In what follows we first show this for the case with  $\Omega = \mathbf{I}_I$ , where  $\mathbf{I}_I$  denotes the  $I$ -dimensional identity matrix, then generalize the result to the case where  $\Omega$  can have arbitrary positive diagonal elements.

For  $\tau > 0$  define hyperplanes

$$H_k^\tau = \{x : b_k' x = -\tau\phi^k\}, 1 \leq k \leq m, \tau > 0,$$

and half spaces

$$H_{\leq k}^\tau(\delta) = \{x : b_k' x \leq \tau\delta\}, 1 \leq k \leq m, \tau > 0,$$

and also let

$$S_k(\delta) = \{x \in C : |b_k' x| \leq \tau\delta\}$$

for  $1 \leq k \leq m$ ,  $\delta > 0$ . In what follows we show that for small enough  $\delta > 0$ , every element  $x^* \in \mathbf{R}^I$  such that

$$(11.2) \quad x^* \in S_1(\delta) \cap \cdots \cap S_q(\delta) \cap H_{\mathcal{L}_{q+1}}^\tau(\delta) \cap \cdots \cap H_{\mathcal{L}_m}^\tau(\delta) \text{ for some } q \in \{1, \dots, m\}$$

satisfies

$$(11.3) \quad x^*|_{C_\tau} \in H_1^\tau \cap \cdots \cap H_q^\tau$$

where  $x^*|_{C_\tau}$  denotes the orthogonal projection of  $x^*$  on  $C_\tau$ . Let  $g^{*k} = b'_k x^*$ ,  $k = 1, \dots, m$ . Note that an element  $x^*$  fulfils (11.2) iff  $|g^{*k}| \leq \tau\delta$ ,  $1 \leq k \leq q$  and  $g^{*j} \leq \tau\delta$ ,  $q+1 \leq j \leq m$ . Likewise, (11.3) holds iff  $\beta_k^\tau(g^*) = 0$ ,  $1 \leq k \leq c$ . Thus in order to establish the desired property of the function  $\beta_\tau(\cdot)$ , we show that (11.2) implies (11.3). Suppose it does not hold; then without loss of generality, for an element  $x^*$  that satisfies (11.2) for an arbitrary small  $\delta > 0$ , we have

$$(11.4) \quad x^*|_{C_\tau} \in H_1^\tau \cap \cdots \cap H_r^\tau \quad \text{and} \quad x^*|_{C_\tau} \notin H_j^\tau, r+1 \leq j \leq q$$

for some  $1 \leq r \leq q-1$ . Define halfspaces

$$H_{\mathcal{L}_k}^\tau = \{x : b'_k x \leq -\tau\phi^k\}, \quad 1 \leq k \leq m, \tau > 0,$$

hyperplanes

$$H_k = \{x : b'_k x = 0\}, 1 \leq k \leq m,$$

and also let

$$F = H_1 \cap \cdots \cap H_r \cap C,$$

then for (11.4) to hold for some  $x^* \in \mathbf{R}^I$  satisfying (11.2) for an arbitrary small  $\delta > 0$  we must have

$$F|(H_1^\tau \cap \cdots \cap H_r^\tau) \subset \text{int}(H_{\mathcal{L}_{r+1}}^\tau \cap \cdots \cap H_{\mathcal{L}_q}^\tau)$$

(Recall the notation  $|$  signifies orthogonal projection. Also note that if  $\dim(F) = 1$ , then  $F = \bigcap_{j=1}^q H_j^\tau$ , and (11.4) does not occur.) Therefore if we let

$$\Delta(J) = \{x \in \mathbf{R}^I : \mathbf{1}'_I x = J, x \geq 0\},$$

i.e. the simplex with vertices  $(J, 0, \dots, 0), \dots, (0, \dots, 0, J)$ , we have

$$(11.5) \quad (F \cap \Delta(J))|(H_1^\tau \cap \cdots \cap H_r^\tau) \subset \text{int}(H_{\mathcal{L}_{r+1}}^\tau \cap \cdots \cap H_{\mathcal{L}_q}^\tau).$$

Let  $\{a_1, \dots, a_H\} = \mathcal{A}$  denote the collection of the column vectors of  $A$ . Then  $\{\text{the vertices of } F \cap \Delta(J)\} \in \mathcal{A}$ . Let  $\bar{a}, \bar{\bar{a}} \in F \cap \Delta(J)$ . Let  $B(\varepsilon, x)$  denote the  $\varepsilon$ -(open) ball with center  $x \in \mathbf{R}^I$ . By (11.5),

$$B(\varepsilon, (\bar{a} | \cap_{j=1}^r H_j^\tau)) \subset \text{int}(H_{\mathcal{L}_{r+1}}^\tau \cap \dots \cap H_{\mathcal{L}_q}^\tau) \cap H_{\mathcal{L}_1} \cap \dots \cap H_{\mathcal{L}_r}$$

holds for small enough  $\varepsilon > 0$ . Let  $\bar{a}^\tau := \bar{a} + \tau$ ,  $\bar{\bar{a}}^\tau := \bar{\bar{a}} + \tau$ , then

$$\begin{aligned} ((\bar{a} | \cap_{j=1}^r H_j^\tau) - \bar{a})'(\bar{\bar{a}} - \bar{a}) &= ((\bar{a} | \cap_{j=1}^r H_j^\tau) - \bar{a})'(\bar{\bar{a}}^\tau - \bar{a}^\tau) \\ &= 0 \end{aligned}$$

since  $\bar{a}^\tau, \bar{\bar{a}}^\tau \in \cap_{j=1}^r H_j^\tau$ . We can then take  $z \in B(\varepsilon, (\bar{a} | \cap_{j=1}^r H_j^\tau))$  such that  $(z - \bar{a})'(\bar{\bar{a}} - \bar{a}) < 0$ . By construction  $z \in C$ , which implies the existence of a triplet  $(a, \bar{a}, \bar{\bar{a}})$  of distinct elements in  $\mathcal{A}$  such that  $(a - \bar{a})'(\bar{\bar{a}} - \bar{a}) < 0$ . In what follows we show that this cannot happen, then the desired property of  $\beta_\tau$  is established.

So let us now show that

$$(11.6) \quad (a_1 - a_0)'(a_2 - a_0) \geq 0 \text{ for every triplet } (a_0, a_1, a_2) \text{ of distinct elements in } \mathcal{A}.$$

Noting that  $a'_i a_j$  just counts the number of budgets on which  $i$  and  $j$  agree, define

$$\phi(a_i, a_j) = J - a'_i a_j,$$

the number of disagreements. Importantly, note that  $\phi(a_i, a_j) = \phi(a_j, a_i)$  and that  $\phi$  is a distance (it is the taxicab distance between elements in  $\mathcal{A}$ , which are all 0-1 vectors). Now

$$\begin{aligned} &(a_1 - a_0)'(a_2 - a_0) \\ &= a'_1 a_2 - a'_0 a_2 - a'_1 a_0 + a'_0 a_0 \\ &= J - \phi(a_1, a_2) - (J - \phi(a_0, a_2)) - (J - \phi(a_0, a_1)) + J \\ &= \phi(a_0, a_2) + \phi(a_0, a_1) - \phi(a_1, a_2) \geq 0 \end{aligned}$$

by the triangle inequality.

Next we treat the case where  $\Omega$  is not necessarily  $\mathbf{I}_I$ . Write

$$\Omega = \begin{bmatrix} \omega_1^2 & 0 & \dots & 0 \\ 0 & \omega_2^2 & \dots & 0 \\ & & \ddots & \\ 0 & \dots & 0 & \omega_I^2 \end{bmatrix}.$$

The statistic  $J_N$  in (3.1) can be rewritten, using the square-root matrix  $\Omega^{1/2}$ ,

$$J_N = \min_{\eta^* = \Omega^{1/2}\eta; \eta \in C} [\hat{\pi}^* - \eta^*]' [\hat{\pi}^* - \eta^*]$$

or

$$J_N = \min_{\eta^* \in C^*} [\hat{\pi}^* - \eta^*]' [\hat{\pi}^* - \eta^*]$$

where

$$\begin{aligned} C^* &= \{\Omega^{1/2}A\nu | \nu \geq 0\} \\ &= \{A^*\nu | \nu \geq 0\} \end{aligned}$$

with

$$A^* = [a_1^*, \dots, a_H^*], a_h^* = \Omega^{1/2}a_h, 1 \leq h \leq H.$$

Then we can follow our previous argument replacing  $a$ 's with  $a^*$ 's, and using

$$\Delta^*(J) = \text{conv}([0, \dots, \omega_i, \dots, 0]' \in \mathbf{R}^I, i = 1, \dots, I).$$

instead of the simplex  $\Delta(J)$ . Finally, we need to verify that the acuteness condition (11.6) holds for  $\mathcal{A}^* = \{a_1^*, \dots, a_H^*\}$ .

For two  $I$ -vectors  $a$  and  $b$ , define a weighted taxicab metric

$$\phi_\Omega(a, b) := \sum_{i=1}^I \omega_i |a_i - b_i|,$$

then the standard taxicab metric  $\phi$  used above is  $\phi_\Omega$  with  $\Omega = \mathbf{I}_I$ . Moreover, letting  $a^* = \Omega^{1/2}a$  and  $b^* = \Omega^{1/2}b$ , where each of  $a$  and  $b$  is an  $I$ -dimensional 0-1 vector, we have

$$a^{*'}b^* = \sum_{i=1}^I \omega_i [1 - |a_i - b_i|] = \bar{\omega} - \phi_\Omega(a, b)$$

with  $\bar{\omega} = \sum_{i=1}^I \omega_i$ . Then for every triplet  $(a_0^*, a_1^*, a_2^*)$  of distinct elements in  $\mathcal{A}^*$

$$\begin{aligned} (a_1^* - a_0^*)'(a_2^* - a_0^*) &= \bar{\omega} - \phi_\Omega(a_1, a_2) - \bar{\omega} + \phi_\Omega(a_0, a_2) - \bar{\omega} + \phi_\Omega(a_0, a_1) + \bar{\omega} - \phi_\Omega(a_0, a_0) \\ &= \phi_\Omega(a_1, a_2) - \phi_\Omega(a_0, a_2) - \phi_\Omega(a_0, a_1) \\ &\geq 0, \end{aligned}$$

which is the desired acuteness condition. Since  $J_N$  can be written as the minimum of the quadratic form with identity-matrix weighting subject to the cone generated by  $a^*$ 's, all the previous arguments developed for the case with  $\Omega = \mathbf{I}_I$  remain valid.

Defining  $\xi \sim N(0, \hat{S})$  and  $\zeta = B\xi$ ,

$$\begin{aligned} \tilde{J}_N(\tau_N) &\sim \min_{Bt \leq -\tau_N \phi} N \left[ \begin{pmatrix} KB \\ M \end{pmatrix} (\hat{\eta}_{\tau_N} + N^{-1/2}\xi - t) \right]' P^{-1'} \Omega P^{-1} \left[ \begin{pmatrix} KB \\ M \end{pmatrix} (\hat{\eta}_{\tau_N} + N^{-1/2}\xi - t) \right] \\ &= N \min_{\gamma \leq -\tau_N \phi, \gamma \in \text{col}(B)} t \left( K \left[ \hat{\gamma}_{\tau_N} + N^{-1/2}\zeta - \gamma \right] \right). \end{aligned}$$

Moreover, defining  $\gamma^\tau = \gamma + \tau_N \phi$  in the above, and using the definitions of  $\beta_\tau(\cdot)$  and  $s(\cdot)$

$$\begin{aligned} \tilde{J}_N(\tau_N) &\sim N \min_{\gamma^\tau \leq 0, \gamma^\tau \in \text{col}(B)} t \left( K \left[ \hat{\gamma}_{\tau_N} + \tau_N \phi + N^{-1/2}\zeta - \gamma^\tau \right] \right) \\ &= N \min_{\gamma^\tau \leq 0, \gamma^\tau \in \text{col}(B)} t \left( K \left[ \gamma_{\tau_N}(\hat{g}) + \tau_N \phi + N^{-1/2}\zeta - \gamma^\tau \right] \right) \\ &= N \min_{\gamma^\tau \leq 0, \gamma^\tau \in \text{col}(B)} t \left( K \left[ \beta_{\tau_N}(\hat{g}) + N^{-1/2}\zeta - \gamma^\tau \right] \right) \\ &= s \left( N^{1/2} \beta_{\tau_N}(\hat{g}) + \zeta \right) \end{aligned}$$

Let  $\varphi_N(\xi) := N^{1/2} \beta_{\tau_N}(\tau_N \xi)$  for  $\xi = (\xi_1, \dots, \xi_m)' \in \text{col}(B)$ , then from the property of  $\beta_\tau$  shown above, its  $k$ -th element  $\varphi_N^k$  satisfies

$$\varphi_N^k(\xi) = 0$$

if  $|\xi^k| \leq \delta$  and  $\xi^j \leq \delta, 1 \leq j \leq m$  for large enough  $N$ . Define  $\hat{\xi} := \hat{g}/\tau_N$  and using the definition of  $\varphi_N$ , we write

$$(11.7) \quad \tilde{J}_N(\tau_N) \sim s \left( \varphi_{\tau_N}(\hat{\xi}) + \zeta \right).$$

Now we invoke Theorem 1 of AS10. As noted before, the function  $t$  is a positive definite quadratic form on  $\mathbf{R}^\ell$ , and so is its restriction on  $\text{col}(B)$ . Then Assumptions 1-3 of AS10 hold for the function  $s$  defined above if signs are adjusted appropriately as our formulae deal with negativity constraints whereas AS10 is formulated for positivity constraints (note that Assumption 1(b) does not apply here since we use a fixed weighting matrix). The function  $\varphi_N$  in (11.7) satisfies the properties of  $\varphi$  in AS10 used in their proof of Theorem 1. AS10 imposes a set of restrictions on the parameter space (see their Equation (2.2) on page 124). Their condition (2.2) (vii) is a Lyapunov condition for a triangular array CLT. Following AS10, consider a sequence of distributions  $\pi_N = [\pi'_{1N}, \dots, \pi'_{jN}]', N = 1, 2, \dots$  in  $\mathcal{P} \cap C$  such that (1)  $\sqrt{N}B\pi_N \rightarrow h$  for a non-positive  $h$  as  $N \rightarrow \infty$  and (2)  $\text{Cov}_{\pi_N}(\sqrt{N}B\hat{\pi}) \rightarrow \Sigma$  as  $N \rightarrow \infty$  where  $\Sigma$  is positive semidefinite. As  $\pi_N \in \mathcal{P}$ , Condition 5.1 demands that  $\text{var}_{\pi_{jN}}(b_k(j)'d_{j,n}) \geq \epsilon$  holds for some  $j$  for each  $N$ . Therefore we have  $\lim_{N \rightarrow \infty} \text{var}_{\pi_{jN}}(b_k(j)'d_{j,n}) \geq \epsilon$  for some  $j$ . For such  $j$ , the Lindeberg condition for CLT holds for  $\sqrt{N(j)}[b_1(j)'[\hat{\pi}_j - \pi_{jN(j)}], \dots, b_m(j)'[\hat{\pi}_j - \pi_{jN(j)}]]'$  where  $\hat{\pi}_j = [\hat{\pi}_{1j}, \dots, \hat{\pi}_{I_j j}]'$ , and we have  $\sqrt{N}B(\hat{\pi} - \pi_N) \stackrel{\pi_N}{\rightsquigarrow} N(0, \Sigma)$ . The other conditions (2.2)(i)-(vi) hold



trivially. Finally, Assumptions GMS 2 and GMS 4 of AS10 are concerned with their thresholding parameter  $\kappa_N$  for the  $k$ -th moment inequality, and by letting  $\kappa_N = N^{1/2}\tau_N\phi_k$ , the former holds by the condition  $\sqrt{N}\tau_N \uparrow \infty$  and the latter by  $\tau_N \downarrow 0$ . Therefore we conclude

$$\liminf_{N \rightarrow \infty} \inf_{\pi \in \mathcal{P} \cap C} \Pr\{J_N \leq \hat{c}_{1-\alpha}\} = 1 - \alpha.$$

□

Before stating the proof of Corollary 7.1, let us introduce some notation.

**Notation.** Let  $B^{(j)} := [b_1(j), \dots, b_k(j)]' \in \mathbf{R}^{k \times I_j}$ . For  $F \in \mathcal{F}$  and  $1 \leq j \leq J$ , define

$$p_F^{(j)}(x) := E_F[d_{j,n}|x_n(j) = x], \quad \pi_F^{(j)} = p_F^{(j)}(\mathbf{x}(j)), \quad \pi_F = [\pi_F^{(1)'}, \dots, \pi_F^{(J)'}]'$$

and

$$\Sigma_F^{(j)}(x) := \text{Cov}_F[d_{j,n}|x_n(j) = x].$$

Also let  $f_F^{(j)}$  denote the density of  $x_n(j)$  under  $F$ .

Note that  $\Sigma_F^{(j)}(x) = \text{diag}(p_F^{(j)}(x)) - p_F^{(j)}(x)p_F^{(j)}(x)'$ . The following is used in the proof of Corollary 7.1:

**Proposition 11.1.** *If Conditions 7.1(iv) and (v) hold for some  $K < \infty$  and  $\rho$  such that  $\rho(t) \rightarrow 0$  as  $t \rightarrow 0$ , then  $\{p_{F \in \mathcal{F}}^{(j)}\}$ ,  $\{\dot{p}_{F \in \mathcal{F}}^{(j)}\}$ ,  $\{\ddot{p}_{F \in \mathcal{F}}^{(j)}\}$ ,  $\{f_{F \in \mathcal{F}}^{(j)}\}$ ,  $\{\dot{f}_{F \in \mathcal{F}}^{(j)}\}$ , and  $\{\ddot{f}_{F \in \mathcal{F}}^{(j)}\}$  are equicontinuous at  $\mathbf{x}(j)$  for all  $1 \leq j \leq J$ .*

The proof is straightforward and thus omitted.

**Proof of Corollary 7.1.** The proof follows the same steps as those in the proof of Theorem 5.1, except for the treatment of  $\hat{\pi}$ . Therefore, instead of the sequence  $\pi_N, N = 1, 2, \dots$  in  $\mathcal{P} \cap C$ , consider a sequence of distributions  $F_N = [F_{1N}, \dots, F_{JN}]$ ,  $N = 1, 2, \dots$  in  $\mathcal{F}$  such that (1)  $\sqrt{N}h_N B \pi_{F_N} \rightarrow h$  for a non-positive  $h$  as  $N \rightarrow \infty$ ; (2)  $\Sigma_{F_N}^{(j)}(\mathbf{x}(j)) \rightarrow \Sigma^{(j)}(\mathbf{x}(j))$  as  $N \rightarrow \infty$ ,  $1 \leq j \leq J$  where  $\Sigma^{(j)}$  is positive semidefinite; and (3)  $f_{F_N}^{(j)}(\mathbf{x}(j)) \rightarrow f^{(j)}(\mathbf{x}(j))$  as  $N \rightarrow \infty$ ,  $1 \leq j \leq J$ .

Define

$$v_{N,n}(j) = \frac{1}{\sqrt{h_N}} K \left( \frac{x_n(j) - \mathbf{x}(j)}{h_N} \right) B^{(j)} [d_{j,n} - p_{F_N}^{(j)}(x_n(j))],$$

then

$$(11.8) \quad E_{F_N}[v_{N,n}(j)] = 0, \quad 1 \leq j \leq J.$$

By the standard change of variable argument

$$\text{Cov}_{F_N}(v_{N,n}(j)) = \int B^{(j)} \Sigma_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) B^{(j)'} f_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) K^2(z) dz$$

for  $1 \leq j \leq J$ . Let  $\Delta := \text{Cov}_{F_N}(v_{N,n}(j)) - B^{(j)} \Sigma^{(j)}(\mathbf{x}(j)) B^{(j)'} \int K^2(z) dz$ , then

$$\begin{aligned} \Delta &= \int B^{(j)} \left[ \Sigma_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) - \Sigma^{(j)}(\mathbf{x}(j)) \right] B^{(j)'} f_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) K^2(z) dz \\ &\quad + B^{(j)} \Sigma_{F_N}^{(j)}(\mathbf{x}(j)) B^{(j)'} \int \left[ f_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) - f^{(j)}(\mathbf{x}(j)) \right] K^2(z) dz, \end{aligned}$$

but as  $N \rightarrow \infty$

$$\begin{aligned} &\left\| \Sigma_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) - \Sigma^{(j)}(\mathbf{x}(j)) \right\| \\ &\leq \left\| \Sigma_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) - \Sigma_{F_N}^{(j)}(\mathbf{x}(j)) \right\| + \left\| \Sigma_{F_N}^{(j)}(\mathbf{x}(j)) - \Sigma^{(j)}(\mathbf{x}(j)) \right\| \\ &\leq \sup_{F \in \mathcal{F}} \left\| \Sigma_F^{(j)}(\mathbf{x}(j) + h_N z) - \Sigma_F^{(j)}(\mathbf{x}(j)) \right\| + \left\| \Sigma_{F_N}^{(j)}(\mathbf{x}(j)) - \Sigma^{(j)}(\mathbf{x}(j)) \right\| \\ &\rightarrow 0 \quad \text{for every } z \end{aligned}$$

where the convergence holds by Proposition 11.1. Similarly, as  $N \rightarrow \infty$

$$\left| f_{F_N}^{(j)}(\mathbf{x}(j) + h_N z) - f^{(j)}(\mathbf{x}(j)) \right| \rightarrow 0 \quad \text{for every } z.$$

By the Bounded Convergence Theorem conclude that

$$(11.9) \quad \Delta \rightarrow 0.$$

This means that a version of equation (2.2.10) in Bierens (1987) still holds even under the sequence of distributions  $\{F_N\}_{N=1}^{\infty}$  as specified above.

We next verify the Lyapunov condition. As  $F_N \in \mathcal{F}$ , Condition 7.1(ii) demands that

$$\text{var}_{F_N}(b_k(j)' d_{j,n} | x_n = \mathbf{x}(j)) \geq \epsilon$$

holds for some  $j$  for each  $N$ . Thus we have  $\lim_{N \rightarrow \infty} \text{var}_{F_N}(b_k(j)' d_{j,n} | x_n = \mathbf{x}(j)) \geq \epsilon$  for some  $j$ . Define  $s_N^2 := \text{Cov}_{F_N} \left( \sum_{n=1}^N v_{N,n}(i|j) \right)$  for such  $j$  where  $v_{N,n}(i|j)$  denotes the  $i$ -th element of the  $I_j$  random vector  $v_{N,n}(j)$ . Note that  $s_N \asymp N_j$ , and therefore proceeding as in the derivation of (11.9), then using Equation (2.2.10) in Bierens (1987) we obtain

$$(11.10) \quad \frac{1}{s_{N_j}^{2+\delta}} \sum_{n=1}^{N_j} E_{F_N} [|v_{N,n}(i|j)|^{2+\delta}] \rightarrow 0$$

for  $1 \leq i \leq I_j$ . By (11.8), (11.9) and (11.10), we have

$$\frac{1}{\sqrt{N_j}} \sum_{n=1}^{N_j} v_{N_j, n}(j) \stackrel{F_N}{\rightsquigarrow} N(0, \Omega_1), \quad \Omega_1 = B^{(j)} \Sigma^{(j)}(\mathbf{x}(j)) B^{(j)'} f^{(j)}(\mathbf{x}(j)) \int K^2(z) dz.$$

Finally, the convergence results corresponding (2.2.4) and (2.2.5) in Bierens (1987) hold under the sequence of distributions  $\{F_N\}_{N=1}^{\infty}$  as specified above, once again by proceeding as in the derivation of (11.9). It follows that

$$\sqrt{N_j h_N} B^{(j)} [\hat{\pi}_j - \pi_{F_N}] \stackrel{F_N}{\rightsquigarrow} N(0, \Omega_2), \quad \Omega_2 = \frac{1}{f^{(j)}(\mathbf{x}(j))} B^{(j)} \Sigma^{(j)}(\mathbf{x}(j)) B^{(j)'} \int K^2(z) dz.$$

The rest is the same as the proof of Theorem 5.1.  $\square$

## 12. APPENDIX B: ALGORITHMS FOR COMPUTING $A$

This appendix algorithms for computation of  $A$ . The first algorithm is a brute-force approach that generates all possible choice patterns and then verifies which of these are rationalizable. The second one avoids the construction of the vast majority of possible choice patterns because it checks for rationality along the way as choice patterns are constructed. The third algorithm uses proposition 1. To give a sense of the algorithms' performance, the matrix  $A$  corresponding to the 1975-1982 data, which is of size  $[50 \times 42625]$  and cannot be computed with our implementation of algorithm 1, computes in about 2 hours with our implementation of algorithm 2, and (after suitable rearrangement of budgets) in about 2 minutes with our implementation of algorithm 3. All implementations are in MATLAB and are available from the authors. The instruction to FW-test a sequence refers to use of the Floyd-Warshall algorithm to detect choice cycles. We use the FastFloyd implementation due to Dustin Arendt (<http://www.mathworks.com/matlabcentral/fileexchange/25776-vectorized-floyd-warshall/content/FastFloyd.m>).

### Algorithm 1: Brute Force

This algorithm is easiest described verbally. First generate a matrix  $A^{\max}$  that contains all logically possible choice patterns. To do so, let  $E^i$  denote the set of unit vectors in  $R^i$  and observe that a stacked vector  $(a'_1, \dots, a'_J)'$  is a column of  $A^{\max}$  iff  $(a_1, \dots, a_J) \in E^{I_1} \times \dots \times E^{I_J}$ . It is then easy to construct  $A^{\max}$  by looping. Next, generate  $A$  by FM-testing every column of  $A^{\max}$  and retaining only columns that pass.

### Algorithm 2: Decision Tree Crawling

An intuition for this algorithm is as follows. All possible choice patterns can be arranged on one decision tree, where the first node refers to choice from  $B_1$  and so forth. The tree is systematically crawled. Exploration of any branch is stopped as soon as a choice cycle is detected. Completion of a rationalizable choice pattern is detected when a terminal node has been reached.

Pseudo-code for this algorithm follows.

1. Initialize  $m_1 = \dots = m_J = 1$ .
2. Initialize  $l = 2$ .
3. Set  $c(B_1) = m_1, \dots, c(B_l) = m_l$ . FW-test  $(c(B_1), \dots, c(B_l))$ .
4. If no cycle is detected, move to step 5. Else:
  - 4a. If  $m_l < I_l$ , set  $m_l = m_l + 1$  and return to step 3.
  - 4b. If  $m_l = I_l$  and  $m_{l-1} < I_{l-1}$ , set  $m_l = 1$ ,  $m_{l-1} = m_{l-1} + 1$ ,  $l = l - 1$ , and return to step 3.
  - 4c. If  $m_l = I_l$ ,  $m_{l-1} = I_{l-1}$ , and  $m_{l-2} < I_{l-2}$ , set  $m_l = m_{l-1} = 1$ ,  $m_{l-2} = m_{l-2} + 1$ ,  $l = l - 2$ , and return to step 3.
  - (...)
  - 4z. Terminate.
5. If  $l < J$ , set  $l = l + 1$ ,  $m_l = 1$ , and return to step 3.
6. Extend  $A$  by the column  $[m_1, \dots, m_J]'$ . Also:
  - 6a. If  $m_J < I_J$ , set  $m_J = m_J + 1$  and return to step 3.
  - 6b. If  $m_J = I_J$  and  $m_{J-1} < I_{J-1}$ , set  $m_J = 1$ ,  $m_{J-1} = m_{J-1} + 1$ ,  $l = J - 1$ , and return to step 3.
  - 6c. If  $m_l = I_l$ ,  $m_{l-1} = I_{l-1}$ , and  $m_{l-2} < I_{l-2}$ , set  $m_l = m_{l-1} = 1$ ,  $m_{l-2} = m_{l-2} + 1$ ,  $l = l - 2$ , and return to step 3.
  - (...)
  - 6z. Terminate.

### Algorithm 3: Refinement using Proposition 4.1

Let budgets be arranged s.t.  $(B_1, \dots, B_M)$  do not intersect  $B_J$ ; for exposition of the algorithm, assume  $B_J$  is above these budgets. Then pseudo-code for an algorithm that exploits proposition 1 (calling either of the preceding algorithms for intermediate steps) is as follows.

1. Use brute force or crawling to compute a matrix  $A_{M+1 \rightarrow J-1}$  corresponding to budgets  $(B_{M+1}, \dots, B_J)$ , though using the full  $X$  corresponding to budgets  $(B_1, \dots, B_J)$ .<sup>15</sup>
2. For each column  $a_{M+1 \rightarrow J-1}$  of  $A_{M+1 \rightarrow J-1}$ , go through the following steps:
  - 2.1 Compute (by brute force or crawling) all vectors  $a_{1 \rightarrow M}$  s.t.  $(a_{1 \rightarrow M}, a_{M+1, J-1})$  is rationalizable.
  - 2.2 Compute (by brute force or crawling) all vectors  $a_J$  s.t.  $(a_{M+1, J-1}, a_J)$  is rationalizable.
  - 2.3 All stacked vectors  $(a'_{1 \rightarrow M}, a'_{M+1, J-1}, a'_J)'$  are valid columns of  $A$ .

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<sup>15</sup>(This matrix has more rows than an  $A$  matrix that is only intended to apply to choice problems  $(B_{M+1}, \dots, B_J)$ .)

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