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Mean-Variance Efficiency and Intertemporal Price for Risk

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**MEAN-VARIANCE EFFICIENCY
AND
INTERTEMPORAL PRICE FOR RISK**

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ABSTRACT. In a continuous time, arbitrage free, non-complete market with a zero bond, we find the intertemporal price for risk to equal the standard deviation of the discounted variance optimal martingale measure divided by the zero bond price. We show the Hedging Numéraire to equal the Market Portfolio and find the mean-variance efficient portfolios.

Keywords: CAPM, Market Portfolio, Sharpe-Ratio, Price for Risk, Mean-Variance Efficient Frontier, Hedging Numéraire.

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INTRODUCTION

Using the equivalent martingale measure approach, see Harrison and Pliska (1981), Delbaen and Schachermayer (1994), we study the problem of finding mean-variance efficient portfolios which is closely related to the notion of a price for risk. Originally, the first probabilistic (single-period) market models (CAPM) were based on the idea of a price for risk and the notion of mean-variance efficiency, see Markowitz (1952, 1987), Sharpe (1964), Lintner (1965), Jensen (1972), Elton and Gruber (1979). See also Li and Ng (2000) for a multi-period model. The notion of a price for risk often appears in connection with the equivalent martingale measure.

In the most general case, where we consider a (not necessarily continuous but locally \mathcal{S}^2) semimartingale market model, the central idea for solving this problem is just the notion of orthogonality, whereas in the case of a continuous semimartingale market model, where we will derive stronger results, the central idea comes from stochastic duality theory, which goes back to Bismut (1973, 1975). For an alternative approach based on BSDE theory, see Zhou and Li (2000), Lim and Zhou (2000).

After some technical preparations in Section 1, we introduce in Section 2 different spaces of equivalent local martingale measures and several spaces of self-financing hedging strategies. In Section 3 we prove the existence of the discounted variance-optimal martingale measure and the hedging numéraire for a locally \mathcal{S}^2 -semimartingale market model, see Gourieroux, Laurent and Pham (1998), (GLP98), for this result in the continuous case.

In Section 4 we generalize the classical single-period results about mean-variance efficient portfolios to the case of a general semimartingale market model. Assuming the existence of zero bonds in the market, the mean-variance efficient portfolios are shown to be linear combinations of the hedging numéraire and the zero bond. We find the mean-variance efficient market line for a fixed time horizon to be a straight line with a slope equal to the standard deviation of the discounted variance-optimal martingale measure divided by the zero bond price. This quantity can be interpreted as a price for intertemporal risk. In Section 5 we develop a conditional version of these results in the case of a continuous semimartingale market model. This allows to define the *term structure of the intertemporal price for risk* and its dynamic. In Section 6 we discuss an application of our results to the problem of pricing non-attainable claims and the relation of the term structure of interest rates to the term structure of the intertemporal price for risk.

1. SELF-FINANCING HEDGING STRATEGIES

Let a filtered probability space $\Omega_\infty := (\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, P)$, satisfying the usual conditions be given. For simplicity we assume \mathcal{F}_0 to be trivial up to sets of measure 0 with respect to P and $\mathcal{F}_{\infty-} = \mathcal{F}_\infty := \mathcal{F}$. For a process X and a map $\tau : \Omega \rightarrow \mathbb{R}_+$, denote the stopped process at time τ by X^τ . We will often restrict a semimartingale X on Ω_∞ to an interval $[t, T]$, $0 \leq t \leq T < \infty$, resp. to $[t, \infty)$. Therefore we introduce the following filtered probability space (again satisfying the usual conditions), $\Omega_{[t, T]} := \left(\Omega, \mathcal{F}_T, \left(\mathcal{F}_s^{[t, T]} \right)_{s \geq 0}, P|_{\mathcal{F}_T} \right)$ for all $0 \leq t \leq T \leq \infty$, $t < \infty$, where $\mathcal{F}_s^{[t, T]} := \mathcal{F}_{t \vee s \wedge T}$ for $0 \leq s < \infty$. The process $X_s^{[t, T]} := X_{t \vee s \wedge T}$ is then a semimartingale on $\Omega_{[t, T]}$. However, on $[t, T]$ we often write X instead of $X^{[t, T]}$. Set $\Omega_T := \Omega_{[0, T]}$.

Define $L^2(\Omega_{[t, T]})$, resp. $L_t^2(\Omega_{[t, T]})$, as the set of \mathcal{F}_T -measurable random variables X , such that $E[X] < \infty$ a.s., resp. $E_t[X] < \infty$ a.s., where $E_t[\cdot] := E[\cdot | \mathcal{F}_t]$ denotes the generalized conditional expectation. The conditional variance is denoted by $\text{Var}_t(\cdot)$.

The stochastic exponential of a semimartingale X is denoted as $\mathcal{E}(X)$. As a general reference we cite Jacod and Shiryaev (1987), (J&S 87) and Jacod (1979). Denote the set of predictable processes which are locally integrable, resp. locally Riemann-Stieltjes integrable, with respect to a local martingale M , resp. with respect to a process A of finite variation, by $L_{loc}^1(M)$, resp. by $L_{loc}^1(A)$. If the semimartingale X admits a decomposition $X = X_0 + A + M$, where M is a local martingale and A is a process of finite variation then $L_{loc}^1(X) := L_{loc}^1(M) \cap L_{loc}^1(A)$.

We can now define the market model: Let $S = (S_t)_{0 \leq t < \infty}$ be a \mathbb{R}^d -valued semimartingale. $\mathcal{M} := (\Omega_\infty, S) = ((\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq 0}, P), S)$ is a model for a market, where S describes the price processes of d assets. We will often consider such a market on an interval $[t, T]$, $0 \leq t < T < \infty$. This is equivalent to work with the following market model $\mathcal{M}_{[t, T]}$ defined by $\mathcal{M}_{[t, T]} := (\Omega_{[t, T]}, S^{[t, T]})$. Set $\mathcal{M}_T := \mathcal{M}_{[0, T]}$.

We want to model the economic activity of investing money into a portfolio of assets and changing the number of assets held over time according to a certain strategy. This is achieved with the following definition:

Definition 1.1. A *hedging strategy* in the market \mathcal{M} is a $H \in L_{loc}^1(S)$.

The corresponding *value process* V^H of H is defined as $V^H := HS$.

The *gains process* of H is defined as the semimartingale $G^H := H \cdot S$. H is called *self-financing* if $V^H = V_0^H + G^H$, i.e. $H_t S_t = H_0 S_0 + \int_0^t H_s dS_s, \forall t \geq 0$. Denote the space of all self-financing hedging strategies in \mathcal{M} by $\mathcal{SF}(\mathcal{M})$.

Note that for $H \in \mathcal{SF}(\mathcal{M})$, we have $H^{[t,T]} \in \mathcal{SF}(\mathcal{M}_{[t,T]})$. The idea of a self-financing hedging strategy is that the changes over time of the corresponding value process are solely caused by the changes of the value of the assets held in the portfolio and not by withdrawing money from or adding money to the portfolio.

Definition 1.2. A semimartingale B such that B and B_- are strictly positive is called a *numéraire* for the market \mathcal{M} . The market discounted with respect to B is then defined as $\mathcal{M}^B := (\Omega_\infty, S^B)$, where $S^B := \frac{S}{B}$. For $0 \leq t \leq T < \infty$, the market restricted to the interval $[t, T]$ is defined as $\mathcal{M}_{[t,T]}^B := (\mathcal{M}^B)_{[t,T]} = (\Omega_{[t,T]}, (S^B)^{[t,T]})$.

Note that for a numéraire B , B^{-1} is a numéraire too and S^B is a semimartingale. The following result is well known, see Geman, El Karoui and Rochet (1995) and Goll and Kallsen (2000):

Proposition 1.3. *Let B be a numéraire for the market \mathcal{M} . Then $\mathcal{SF}(\mathcal{M}^B) = \mathcal{SF}(\mathcal{M})$ holds.*

2. ARBITRAGE-FREE MARKETS

We consider in this section the market $\bar{\mathcal{M}} := (\Omega_\infty, \bar{S})$, where $\bar{S} := (S, B)$ is $\mathbb{R}^d \times \mathbb{R}$ -valued and B is a numéraire, with $B_0 = 1$, which we assume to be uniformly bounded and uniformly bounded away from 0 on finite intervals. For $0 \leq t \leq T \leq \infty, t < \infty$, denote the set of uniformly integrable, resp. local martingales, living on $\Omega_{[t,T]}$ by

$\mathcal{L}^u(\Omega_{[t,T]})$, resp. by $\mathcal{L}(\Omega_{[t,T]})$. Working in the market $\bar{\mathcal{M}}_{[t,T]}$ or on $\Omega_{[t,T]}$ the process $\left(\frac{S}{B}\right)^{[t,T]}$ will be denoted as S^* . Define the following sets of local martingale measures:

$$(2.1) \quad \mathcal{D}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ Z \in \mathcal{L}(\Omega_{[t,T]}) \mid Z \mathbf{1}_{[0,t]} = 1, Z > 0, S^* Z \in \mathcal{L}(\Omega_{[t,T]}) \right\},$$

$$(2.2) \quad \mathcal{D}^e(\bar{\mathcal{M}}_{[t,T]}) := \left\{ Z \in \mathcal{D}(\bar{\mathcal{M}}_{[t,T]}) \mid Z \text{ uniformly integrable martingale} \right\},$$

and

$$(2.3) \quad \mathcal{D}^2(\bar{\mathcal{M}}_{[0,T]}) := \left\{ Z \in \mathcal{D}(\bar{\mathcal{M}}_{[0,T]}) \mid Z_T \in L^2(\Omega_{[0,T]}) \right\},$$

$$(2.4) \quad \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{Z_{tV}}{Z_t} \mid Z \in \mathcal{D}^2(\bar{\mathcal{M}}_{[0,T]}) \right\}.$$

We will now introduce a space of simple self-financing hedging strategies, following closely the ideas of Delbaen and Schachermayer (1996a, 1996b), (DS96a, DS96b). Assume S^* to be locally in $L^2(\Omega_{[0,T]})$ in the sense, that there exists a sequence $U_n, n \in \mathbf{N}$ of localizing stopping times increasing to infinity such that for each $n \in \mathbf{N}$, the family $\{S_\tau \mid \tau \text{ stopping time, } \tau \leq U_n\}$ is bounded in $L^2(\Omega_{[0,T]})$. This condition is certainly satisfied for locally bounded or continuous S . Denote by $\mathcal{H}(\Omega_{[t,T]})$ the set of \mathbb{R}^d -valued predictable processes which are a linear combination of processes of the form $H = h \mathbf{1}_{] \tau_1, \tau_2]}$, where $t \leq \tau_1 \leq \tau_2 \leq T$ are stopping times dominated by some U_n and $h \mathbf{1}_{\{\tau_1 < \tau_2\}}$ is a bounded \mathcal{F}_{τ_1} -measurable random variable. Define the following space of semimartingales:

$$(2.5) \quad \mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \cdot S^* \mid H \in \mathcal{H}(\Omega_{[t,T]}) \right\},$$

and the corresponding space of terminal values

$$(2.6) \quad \mathcal{K}_T^*(\bar{\mathcal{M}}_{[t,T]}) := \left\{ V_T \mid V \in \mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]}) \right\} \subseteq L^2(\Omega_{[t,T]}).$$

Every $H \in \mathcal{H}(\Omega_{[t,T]})$ can be extended to an $\hat{H} \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^{\hat{H}} = 0$, hence $\mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]})$ is just the space of discounted gains processes $\frac{G^{\hat{H}}}{B^{[t,T]}}$ for such \hat{H} . For $H = h \mathbf{1}_{] \tau_1, \tau_2]} \in \mathcal{H}(\Omega_{[t,T]})$ as above, we find $H \cdot S^* = h(S^{*\tau_2} - S^{*\tau_1})$ and for a stopping time τ we have $(H \cdot S^*)^\tau = h(S^{*\tau_2 \wedge \tau} - S^{*\tau_1 \wedge \tau}) = \hat{H} \cdot S^*$, where $\hat{H} := h \mathbf{1}_{] \tau_1 \wedge \tau, \tau_2 \wedge \tau]} \in \mathcal{H}(\Omega_{[t,T]})$, since $h \mathbf{1}_{\{\tau_1 \wedge \tau < \tau_2 \wedge \tau\}} = h \mathbf{1}_{\{\tau_1 < \tau_2\}} \mathbf{1}_{\{\tau_1 < \tau\}}$ is $\mathcal{F}_{\tau_1 \wedge \tau}$ -measurable. Hence $\mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]})$ is stable under stopping. Define the following space of uniformly integrable martingales:

$$(2.7) \quad \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ E[Z \mid \mathcal{F}] \mid Z \in L^2(\Omega_{[t,T]}), E[Z] = 1, \right. \\ \left. E[V_T Z] = 0, \forall V \in \mathcal{K}^*(\bar{\mathcal{M}}_{[0,T]}) \right\},$$

and the corresponding space of terminal values

$$(2.8) \quad \mathcal{D}_T^{s,2}(\bar{\mathcal{M}}_{[t,T]}) := \{Z_T | Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})\}.$$

Note that $\mathcal{D}_T^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ is closed. Since $\mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]})$ is stable under stopping, we find for $V \in \mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]})$ and $Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ and a stopping time $0 \leq \tau < \infty$, $E[V_T Z_T] = 0 = E[V_T^* Z_T] = E[E[V_\tau Z_T | \mathcal{F}_\tau]] = E[V_\tau Z_\tau]$, hence VZ is a uniformly integrable martingale, see J&S 87, Lemma I.1.44. Let $Z \in L^2(\Omega_{[t,T]})$ with $E[Z] = 1$ be such that S^*Z is a local martingale. We want to show that VZ is a uniformly integrable martingale for all $V \in \mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]})$. For this it suffices to show that $S^{*\tau_1}Z$ is uniformly integrable for every stopping time τ_1 bounded by some U_n . We show uniform integrability of the family $\{S_\tau^{*\tau_1} Z_\tau | \tau \text{ stopping time}\}$. We have

$$\begin{aligned} \lim_{K \rightarrow \infty} E \left[\mathbf{1}_{\{|S_\tau^{*\tau_1} Z_\tau| > K\}} \right] &\leq \lim_{K \rightarrow \infty} E \left[\mathbf{1}_{\{|S_\tau^{*\tau_1}| > \sqrt{K}\}} \right] + E \left[\mathbf{1}_{\{|Z_\tau| > \sqrt{K}\}} \right] \\ &= \lim_{K \rightarrow \infty} E \left[\mathbf{1}_{\{|S_\tau^{*\tau_1}|^2 > K\}} \right] = 0, \end{aligned}$$

since $S_\tau^{*\tau_1} = S_{\tau \wedge \tau_1}^*$, $\tau \wedge \tau_1 \leq U_n$ and by the assumed boundedness of $\{S_\tau | \tau \text{ stopping time}, \tau \leq U_n\}$ in $L^2(\Omega_{[0,T]})$. Conversely, for $Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$, it is easily seen that S^*Z is a local martingale. Therefore $\mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ equals the set of *signed* local martingale measures for the market $\bar{\mathcal{M}}_{[t,T]}$.

We will work with the following No-Arbitrage condition:

$$(2.9) \quad \mathcal{D}^2(\bar{\mathcal{M}}_T) \neq \emptyset, \quad \forall T < \infty.$$

It implies that

$$(2.10) \quad \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset, \quad \forall 0 \leq t \leq T < \infty.$$

Note that for $Z \in \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]})$ and $t \leq t' \leq T' \leq T$, we have $\frac{Z^{[t',T']}}{Z_{t'}} \in \mathcal{D}_{t'}^2(\bar{\mathcal{M}}_{[t',T']})$. Note also that $\mathcal{D}_0^2(\bar{\mathcal{M}}_{[0,T]}) = \mathcal{D}(\bar{\mathcal{M}}_{[0,T]})$, since \mathcal{F}_0 was assumed to be trivial.

Let $\mathcal{B} \subseteq \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]})$. We call a $H \in \mathcal{B}$ an \mathcal{B} -arbitrage, if $V_0^H = 0$, $V_T^H \geq 0$ and $V_T^H \neq 0$ almost surely. If there exists no \mathcal{B} -arbitrage, then \mathcal{B} is called arbitrage-free. In all probabilistic theories of financial markets allowing to trade at an infinitely large number of instances of time one has to exclude certain self-financing hedging strategies, e.g. doubling strategies, in order to avoid arbitrage opportunities. We will define several arbitrage-free subsets of $\mathcal{SF}(\bar{\mathcal{M}}_{[t,T]})$:

1.

$$(2.11) \quad \mathcal{SF}^b(\bar{\mathcal{M}}_{[t,T]}) := \{H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) | \exists K \in \mathbb{R} : V^H \geq K\}.$$

Let $H \in \mathcal{SF}^b(\bar{\mathcal{M}}_{[t,T]})$, $V_0^H = 0$ and $V^H \geq K$. $\frac{V^H}{B^{[t,T]}}Z$ is then a supermartingale for all $Z \in \mathcal{D}^e(\bar{\mathcal{M}}_{[t,T]})$ and $V_T^H \geq 0$ implies

$$E_t \left[\frac{V_T^H}{B_T} Z_T \right] \leq \frac{V_t^H}{B_t} = 0, \text{ hence } V_T^H = 0.$$

2. For $\mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$, (see DS96b):

$$(2.12) \quad \mathcal{SF}^2(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) \mid V_T^H \in L^2(\Omega_{[t,T]}), \right. \\ \left. \frac{V^H}{B^{[t,T]}}Z \in \mathcal{L}^u(\Omega_{[t,T]}), \forall Z \in \mathcal{D}^2(\bar{\mathcal{M}}_{[t,T]}) \right\},$$

resp.

$$(2.13) \quad \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) \mid V_T^H \in L_t^2(\Omega_{[t,T]}), \right. \\ \left. \frac{V^H}{B^{[t,T]}}Z \in \mathcal{L}^u(\Omega_{[t,T]}), \forall Z \in \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}) \right\}.$$

3.

$$(2.14) \quad \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[0,T]}) \mid V_T^H \in L^2(\Omega_{[t,T]}), \right. \\ \left. \frac{V^H}{B^{[t,T]}}Z \in \mathcal{L}^u(\Omega_{[t,T]}), \forall Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) \right\}.$$

4.

$$(2.15) \quad \mathcal{SF}^0(\bar{\mathcal{M}}_{[t,T]}) := \mathcal{SF}_t^0(\bar{\mathcal{M}}_{[t,T]}) := \{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) \mid V^H \geq 0 \}.$$

5. For $\mathcal{D}^2(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$

$$(2.16) \quad \mathcal{SF}_t^{sup,2}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) \mid E_s \left[\frac{V_T^H}{B_T} Z_T \right] \leq \frac{V_s^H}{B_s} Z_s, \right. \\ \left. \forall t \leq s \leq T, Z \in \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}) \right\}.$$

6. For $\mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[0,T]}) \neq \emptyset$

$$(2.17) \quad \mathcal{SF}^{sup,s,2}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) \mid E_s \left[\frac{V_T^H}{B_T} Z_T \right] \leq \frac{V_s^H}{B_s} Z_s, \right. \\ \left. \forall t \leq s \leq T, Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) \right\}.$$

7. For $\bar{S} \in \mathcal{S}_{loc}^2(\Omega_{[t,T]})$

$$(2.18) \quad \mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]}) := \{H \in \mathcal{SF}(\bar{\mathcal{M}}_{[t,T]}) \mid V^H \in \mathcal{S}^2(\Omega_{[t,T]})\},$$

where $\mathcal{S}^2(\Omega_{[t,T]})$ denotes the space of L^2 -integrable semimartingales, see Delbaen, Monat, Schachermayer, Schweizer and Stricker (1997) (DMSSS97).

Define for \mathcal{F}_t -measurable v

$$(2.19) \quad \mathcal{A}_v^{*2}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{V_T^H}{B_T} \mid H \in \mathcal{SF}^2(\bar{\mathcal{M}}_{[t,T]}), \frac{V_t^H}{B_t} = v \right\},$$

and

$$(2.20) \quad \mathcal{K}_v^{*2}(\bar{\mathcal{M}}_{[t,T]}) := \left\{ \frac{V_T^H}{B_T} \mid H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]}), \frac{V_t^H}{B_t} = v \right\}.$$

Denote by $\bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]})$ the closure of $\mathcal{K}_T^*(\bar{\mathcal{M}}_{[t,T]})$ in $L^2(\Omega_{[0,T]})$ and define

$$(2.21) \quad \bar{\mathcal{A}}_T^*(\bar{\mathcal{M}}_{[t,T]}) := \overline{\mathcal{K}_T^*(\bar{\mathcal{M}}_{[t,T]}) - L_+^2(\Omega_{[t,T]})} \cap \overline{\mathcal{K}_T^*(\bar{\mathcal{M}}_{[t,T]}) + L_+^2(\Omega_{[t,T]})}.$$

In DS96b, Theorem 1.2, Theorem 2.2, the following was shown using a results from Yor (1978):

Theorem 2.1. *Under the above assumptions we have*

$$\begin{aligned} \bar{\mathcal{A}}_T^*(\bar{\mathcal{M}}_{[t,T]}) &= \mathcal{A}_0^{*2}(\bar{\mathcal{M}}_{[t,T]}) \\ &= \{V \in L^2(\Omega_{[t,T]}) \mid E[VZ] = 0, \forall Z \in \mathcal{D}^2(\bar{\mathcal{M}}_{[t,T]})\}. \end{aligned}$$

*In particular, $\mathcal{A}_0^{*2}(\bar{\mathcal{M}}_{[t,T]})$ is closed in $L^2(\Omega_{[t,T]})$. Furthermore, for continuous \bar{S} , we have $\bar{\mathcal{A}}_T^*(\bar{\mathcal{M}}_{[t,T]}) = \bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]})$.*

We will prove the following corollary:

Corollary 2.2. *Under the above assumptions we have*

$$\begin{aligned} \bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]}) &= \mathcal{K}_0^{*2}(\bar{\mathcal{M}}_{[t,T]}) \\ &= \{V \in L^2(\Omega_{[t,T]}) \mid E[VZ_T] = 0, \forall Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})\}. \end{aligned}$$

Proof. For $V \in \bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]})$ choose a sequence $V^n \in \mathcal{K}_T^*(\bar{\mathcal{M}}_{[t,T]})$ converging to V in $L^2(\Omega_{[t,T]})$. Since $E[V^n Z_T] = 0, \forall Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ and since the inclusion $L^2(\Omega_{[t,T]}) \hookrightarrow L^1(\Omega_{[t,T]}, Q^Z)$, where $Z_T = \frac{dQ^Z}{dP}$, is continuous for $Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ by Cauchy-Schwarz inequality, we find $E[V Z_T] = 0, \forall Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$. Conversely, if $V \in L^2(\Omega_{[t,T]})$ and $E[V Z_T] = 0, \forall Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ and $V \notin \bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]})$, then $V \notin \text{span}\{\bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]}), \mathbf{1}\}$, hence by the Hahn-Banach theorem and since $\mathbf{1} \notin \bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]})$, we find an $Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ such that $E[V Z_T] \neq 0$, a contradiction. Since $\bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]}) \subseteq \bar{\mathcal{A}}_T^*(\bar{\mathcal{M}}_{[t,T]})$, we find by Theorem 2.1 for $V \in \bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]})$ a $H \in \mathcal{SF}^2(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^H = 0$ and $\frac{V_T^H}{B_T} = V$. We also find a sequence $\tilde{V}^n \in \mathcal{K}_T^*(\bar{\mathcal{M}}_{[t,T]})$ converging to V in $L^2(\Omega_{[t,T]})$ and a sequence $V^n \in \mathcal{K}^*(\bar{\mathcal{M}}_{[t,T]})$ with $V_T^n = \tilde{V}^n$. For $Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ and $0 \leq s \leq T$ we have for $n, m \rightarrow \infty$

$$\begin{aligned}
E[|V_s^n Z_s - V_s^m Z_s|] &= E[|E[V_T^n Z_T | \mathcal{F}_s] - E[V_T^m Z_T | \mathcal{F}_s]|] \\
&\leq E[E[|V_T^n Z_T - V_T^m Z_T| | \mathcal{F}_s]] \\
&= E[|V_T^n Z_T - V_T^m Z_T|] \rightarrow 0,
\end{aligned}$$

hence $V_s^n Z_s$ is a Cauchy-sequence in $L^1(\Omega_{[t,T]})$. For $\tilde{Z} \in \mathcal{D}^2(\bar{\mathcal{M}}_{[t,T]})$ and $0 \leq s \leq T$, we therefore know that V_s^n is a Cauchy-sequence in $L^1(\Omega_{[t,T]}, Q^{\tilde{Z}})$ converging to V_s^H and doing so $Q^{\tilde{Z}}$ -a.s. and P -a.s.

pointwise for a subsequence. Since for $n \rightarrow \infty$

$$\begin{aligned} E[|V_s^n Z_s - E[V Z_T | \mathcal{F}_s]|] &= E[|E[V_T^n Z_T | \mathcal{F}_s] - E[V Z_T | \mathcal{F}_s]|] \\ &\leq E[E[|V_T^n Z_T - V Z_T| | \mathcal{F}_s]] \\ &= E[|V_T^n Z_T - V Z_T|] \rightarrow 0, \end{aligned}$$

we find $V_s^n Z_s \rightarrow V_s^H Z_s$, hence $E[V Z_T | \mathcal{F}_s] = E[V_T^H Z_T | \mathcal{F}_s] = V_s^H Z_s$.

This proves $\bar{\mathcal{K}}_T^*(\bar{\mathcal{M}}_{[t,T]}) \subseteq \mathcal{K}_0^{*2}(\bar{\mathcal{M}}_{[t,T]})$. The corollary follows now from the obvious inclusion

$$\mathcal{K}_0^{*2}(\bar{\mathcal{M}}_{[t,T]}) \subseteq \{V \in L^2(\Omega_{[t,T]}) | E[V Z_T] = 0, \forall Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})\},$$

and the first part of the proof. \square

We have the following easy result:

Lemma 2.3. *Assume $\mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$ and \bar{S} to be continuous. Then*

$\mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]}) \subseteq \mathcal{S}\mathcal{F}_t^2(\bar{\mathcal{M}}_{[t,T]})$. In particular $\mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]})$ is arbitrage-free.

3. THE DISCOUNTED VARIANCE-OPTIMAL MARTINGALE MEASURE

We will need the discounted variance-optimal martingale measure, introduced in GPL98 for the case of a continuous price process \bar{S} , in the locally \mathcal{S}^2 -semimartingale setting. We generalize the proof of Lemma 2.1. part (c), in DS96a to our situation:

Lemma 3.1. *Assume $\mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) \neq \emptyset$. Then there exists a unique*

element $Z^{opt,t,T} \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ such that $\frac{Z_T^{opt*,t,T}}{B_T} \in B_T \mathcal{K}_{v^t,T}^{*2}(\bar{\mathcal{M}}_{[t,T]})$,*

where

$$(3.1) \quad v^{t,T} = E \left[\left(\frac{Z_T^{opt*,t,T}}{B_T} \right)^2 \right] = \inf_{Z \in \mathcal{D}^{s,2}(\bar{\mathcal{M}}_{[t,T]})} E \left[\left(\frac{Z_T}{B_T} \right)^2 \right] > 0.$$

Furthermore, there exists a $H^{opt} \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ with corresponding value process $V^{opt,t,T} := V^{H^{opt}}$ such that $V_0^{opt,t,T} = 1$,

$$(3.2) \quad V_T^{opt,t,T} = \frac{Z_T^{opt*,t,T}}{v^{t,T} B_t B_T}$$

and

$$(3.3) \quad E \left[\left(V_T^{opt,t,T} \right)^2 \right] = \inf_{V \in B_T \mathcal{K}_{B_t^{-1}}^{s,2}(\bar{\mathcal{M}}_{[t,T]})} E [V^2].$$

Proof. By the uniform boundedness of B and B^{-1} on $[t, T]$, we find the sets $\mathcal{D} := B_T^{-1} \mathcal{D}_T^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ and $\mathcal{K} := B_T \text{span} \{ \mathcal{K}_0^{*2}(\bar{\mathcal{M}}_{[t,T]}), \mathbf{1} \}$ to be closed in $L^2(\Omega_{[t,T]})$. Therefore we find a $\frac{Z^{min}}{B_T} \in \mathcal{D}$ with minimal norm and a representation $\frac{Z^{min}}{B_T} = Z_1 + Z_2$, where $Z_1 \in \mathcal{K}$ and $Z_2 \in \mathcal{K}^\perp$, since $L^2(\Omega_{[t,T]}) \cong \mathcal{K} \oplus \mathcal{K}^\perp$. Denote by $\langle \cdot, \cdot \rangle$ the standard linear product of the Hilbert-space $L^2(\Omega_{[t,T]})$. From $\langle Z_2, \mathcal{K} \rangle = 0$ it follows $E[Z_1 B_T] = 1$ and $\langle Z_1, B_T \mathcal{K}_0^{*2}(\bar{\mathcal{M}}_{[t,T]}) \rangle = 0$, thus $Z_1 \in \mathcal{D}$. Since $\left\| \frac{Z^{min}}{B_T} \right\|^2 = \|Z_1\|^2 + \|Z_2\|^2$ was minimal it follows $Z_2 = 0$, hence $\frac{Z^{min}}{B_T} \in \mathcal{K}$, i.e. there exists a $H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ such that $V_T^H = \frac{Z^{min}}{B_T}$. Define $Z^{opt*,t,T} := E[Z^{min} | \mathcal{F}]$. Uniqueness follows from the strict convexity of $\|\cdot\|^2$. We have $E_t \left[\left(\frac{Z_T^{opt*,t,T}}{B_T} \right)^2 \right] = E_t \left[\frac{V_T^H}{B_T} Z_T^{opt*,t,T} \right] = \frac{V_t^H}{B_t} Z_t^{opt*,t,T}$. By construction $v^{t,T} := \frac{V_t^H}{B_t}$ is deterministic, hence (3.1) follows. Set $H^{opt} := \frac{H}{v^{t,T} B_t}$ and $V^{opt,t,T} := V^{H^{opt}}$. Since $V_0^H = v^{t,T} B_t$,

we have $V_0^{opt,t,T} = 1$. Let $H' \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]})$ such that $V_0^{H'} = 1$. Then $E \left[V_T^{opt,t,T} (V_T^{opt,t,T} - V_T^{H'}) \right] = 0$, since $\frac{V_T^{opt,t,T} - V_T^{H'}}{B_T} \in \mathcal{K}_0^{*2}(\bar{\mathcal{M}}_{[t,T]})$ and $\frac{B_T V_T^{opt,t,T}}{E[B_T V_T^{opt,t,T}]} \in \mathcal{D}_T^{s,2}(\bar{\mathcal{M}}_{[t,T]})$. Therefore $V_T^{opt,t,T}$ is the element with minimal norm in $B_T \mathcal{K}_{B_t}^{*2}(\bar{\mathcal{M}}_{[t,T]})$. \square

Remark 3.2. $V^{opt,t,T}$ is known as the hedging numéraire, see Gourieroux, Laurent and Pham (1998), (GLP98).

4. MEAN-VARIANCE EFFICIENCY

In this section we introduce a first version of the constraint optimization problem known as the Mean-Variance Efficiency problem for the market $\bar{\mathcal{M}}_{[t,T]}$, where \bar{S} is only assumed to be locally in $L^2(\Omega_{[t,T]})$, as described in Section 2.

Define $\mathcal{K} := B_T \mathcal{K}_{B_t}^{*2}(\bar{\mathcal{M}}_{[t,T]})$ and consider the optimization problem

$$(4.1) \quad \mathcal{V}(t, T, e) := \inf_{\substack{H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) \\ V_0^H = 1}} E \left[(V_T^H)^2 \right] = \inf_{V \in \mathcal{K}} E [V^2],$$

under the constraint

$$(4.2) \quad E [V_T^H] = e,$$

for $e \in \mathbb{R}$. Since \mathcal{K} is closed in $L^2(\Omega_{[t,T]})$ by Corollary 2.2 and by strict convexity, there exists a unique $V^{t,T,e} \in \mathcal{K}$ with $\mathcal{V}(t, T, e) = E \left[(V_T^{t,T,e})^2 \right]$ and $E [V_T^{t,T,e}] = e$ iff $\mathcal{K} \cap \{f \in L^2(\Omega_{[t,T]}) | E[f] = e\} \neq \emptyset$. By Lemma 3.1, we have $V^{t,T,\hat{e}^{t,T}} = V^{opt,t,T}$, where $\hat{e} := \hat{e}^{t,T} := E [V_T^{opt,t,T}]$. We call $V^{t,T,e}, e \in \mathbb{R}$ the *mean-variance efficient frontier*. We will prove the following

Proposition 4.1. *Assume the existence of an $V \in \mathcal{K}$ with $E[V] \neq \hat{e}^{t,T}$. Then*

$$(4.3) \quad \mathcal{V}(t, T, e) = \mathcal{V}(t, T, \hat{e}) + c^{t,T} (e - \hat{e})^2, \quad \forall e \in \mathbb{R},$$

for a constant $c^{t,T} \geq 1$. $c^{t,T} = 1$ implies $\hat{e} = 0$ and $\text{Var}(V^{t,T,e}) = \mathcal{V}(t, T, \hat{e})$ for all $e \in \mathbb{R}$.

Furthermore, given $V^{t,T,\bar{e}}$ for some $\bar{e} \neq \hat{e}$ we have

$$(4.4) \quad V^{t,T,e} = s(e)V^{t,T,\hat{e}} + (1-s(e))V^{t,T,\bar{e}}, \quad \forall e \in \mathbb{R},$$

where $s(e) := \frac{e-\bar{e}}{\hat{e}-\bar{e}}$ is defined in such a way that $s(e)\hat{e} + (1-s(e))\bar{e} = e$ holds and

$$(4.5) \quad c^{t,T} = \frac{\mathcal{V}(t, T, \bar{e}) - \mathcal{V}(t, T, \hat{e})}{(\hat{e} - \bar{e})^2}$$

Proof. Let $V \in \mathcal{K}$ with $E[V] \neq \hat{e}$ be given. Since $sV + (1-s)V_T^{opt,t,T} \in \mathcal{K}$ for all $s \in \mathbb{R}$, we find $\mathcal{K} \cap \{f \in L^2(\Omega_{[t,T]}) | E[f] = e\} \neq \emptyset$ for all $e \in \mathbb{R}$, hence $V^{t,T,e}$ exists for all $e \in \mathbb{R}$. Define for $\hat{e} \neq \bar{e}$, and $s \in \mathbb{R}$, $\tilde{\mathcal{V}}_{\bar{e}}(s) := E \left[(sV^{t,T,\hat{e}} + (1-s)V^{t,T,\bar{e}})^2 \right]$. Since $E \left[V_T^{t,T,\hat{e}} V \right] = E \left[\left(V_T^{t,T,\hat{e}} \right)^2 \right]$ for all $V \in \mathcal{K}$ we have $\tilde{\mathcal{V}}_{\bar{e}}(s) = (1 - (1-s)^2) \mathcal{V}(t, T, \hat{e}) + (1-s)^2 \mathcal{V}(t, T, \bar{e})$. Set $\mathcal{V}_{\bar{e}}(e) := \tilde{\mathcal{V}}_{\bar{e}}(s(e))$. We find

$$(4.6) \quad \mathcal{V}_{\bar{e}}(e) = \mathcal{V}(t, T, \hat{e}) + \frac{\mathcal{V}(t, T, \bar{e}) - \mathcal{V}(t, T, \hat{e})}{(\hat{e} - \bar{e})^2} (e - \hat{e})^2.$$

$\mathcal{V}_{\bar{e}}(e)$ is clearly a polynomial of at most second order in e with a minimum of $\mathcal{V}_{\bar{e}}(\hat{e}) = \mathcal{V}(t, T, \hat{e})$ in \hat{e} . The assertion follows now if we show $\mathcal{V}_{\bar{e}}(e) = \mathcal{V}(t, T, e)$ for all $e \in \mathbb{R}$. Since $E \left[s(e)V_T^{t,T,\hat{e}} + (1-s(e))V_T^{t,T,\bar{e}} \right] = s(e)\hat{e} + (1-s(e))\bar{e} = e$ we have $\mathcal{V}_{\bar{e}}(e) \geq \mathcal{V}(t, T, e)$ and $\mathcal{V}_{\bar{e}}(\bar{e}) \geq \mathcal{V}(t, T, \bar{e})$. By a simple calculation, these two inequalities imply $\mathcal{V}_{\bar{e}}(e) = \mathcal{V}(t, T, e)$ for all e, \bar{e} . Calculating $\text{Var}(V_T^{t,T,e}) = \mathcal{V}(t, T, e) - e^2$, which

must be non-negative, we find $c^{t,T} := \frac{\mathcal{V}(t,T,\bar{e}) - \mathcal{V}(t,T,\hat{e})}{(\hat{e} - \bar{e})^2} \geq 1$ and $c^{t,T} = 1$

to imply $\hat{e} = 0$. \square

This result also allows to calculate variance optimal portfolios. Consider the optimization problem

$$(4.7) \quad \tilde{\mathcal{V}}(t, T, e) := \inf_{\substack{H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[t,T]}) \\ v_0^H = 1}} E[\text{Var}(V_T^H)] = \inf_{V \in \mathcal{K}} E[\text{Var}(V)],$$

under the constraint

$$(4.8) \quad E[V_T^H] = e,$$

for $e \in \mathbb{R}$. Since $\tilde{\mathcal{V}}(t, T, e) = \mathcal{V}(t, T, e) - e^2$, we find for $c^{t,T} \neq 1$,

$$(4.9) \quad \min_{e \in \mathbb{R}} \tilde{\mathcal{V}}(t, T, e) = \tilde{\mathcal{V}}\left(t, T, \frac{c^{t,T}}{c^{t,T} - 1} \hat{e}\right) = \mathcal{V}(t, T, \hat{e}) - \frac{c^{t,T}}{c^{t,T} - 1} \hat{e}^2.$$

$c^{t,T} = 1$ implies $\tilde{\mathcal{V}}(t, T, e) = \tilde{\mathcal{V}}(t, T, \hat{e})$ for all $e \in \mathbb{R}$.

Assume now the zero bond B^T with maturity T to be attainable in $\bar{\mathcal{M}}_{[0,T]}$, i.e. there exists a $H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[0,T]})$ such that for $B^T := V^H$, $B_T^T = 1$ holds. This is equivalent to the existence of an almost surely deterministic element in \mathcal{K} . Necessarily we have $B^T > 0$ and B^T is uniformly bounded. The existence of B^T together with the existence of a $V \in \mathcal{K}$ with $E[V] \neq \hat{e}$ implies $c^{0,T} > 1$, since

$$(4.10) \quad \hat{e} = E\left[\frac{Z^{opt*,0,T}}{v^{0,T} B_T}\right] = \frac{E\left[\frac{B_T^T}{B_T} Z^{opt*,0,T}\right]}{v^{0,T}} = \frac{B_0^T}{v^{0,T}} > 0.$$

Hence equation (4.9) implies $\mathcal{V}(0, T, \hat{e}) - \frac{c^{0,T}}{c^{0,T} - 1} \hat{e}^2 = 0$ which is equivalent to $c^{0,T}(\mathcal{V}(0, T, \hat{e}) - \hat{e}^2) = \mathcal{V}(0, T, \hat{e})$. Since $\mathcal{V}(0, T, \hat{e}) > 0$ we find $c^{0,T} = \frac{\mathcal{V}(0, T, \hat{e})}{\text{Var}(V_T^{0,T,\hat{e}})}$. By (3.3), we have

$$(4.11) \quad \mathcal{V}(0, T, \hat{e}) = E\left[\left(V_T^{opt,0,T}\right)^2\right] = \frac{1}{v^{0,T}},$$

hence

$$(4.12) \quad c^{0,T} = \frac{v^{0,T}}{v^{0,T} - (B_0^T)^2}.$$

The unique risk-free self-financing hedging strategy with initial value 1 is just given by $V := \frac{1}{B_0^T} B^T$ and the risk-free return is $V_T = \frac{1}{B_0^T}$. Similar as in the Markowitz single period model we can consider the

ratio of excess expected return e over the risk-free return $\frac{1}{B_0^T}$ and the standard deviation of the return, for $e \neq \frac{1}{B_0^T}$:

$$(4.13) \quad \beta_{[0,T]}(e) := \frac{\left|e - \frac{1}{B_0^T}\right|}{\sqrt{\text{Var}\left(V_T^{0,T,e}\right)}} = \max_{\substack{V \in \mathcal{K} \\ E[V]=e}} \frac{\left|e - \frac{1}{B_0^T}\right|}{\sqrt{\text{Var}(V)}}.$$

Lemma 4.2. *Under the assumption of Proposition 4.1 and assuming the existence of B^T , we have for all $e \neq \frac{1}{B_0^T}$:*

$$(4.14) \quad \beta_{[0,T]}(e) = \frac{\sqrt{\text{Var}\left(\frac{Z^{opt*,0,T}}{B_T}\right)}}{B_0^T} > 0.$$

Proof. The assertion follows from an elementary calculation using formulas (4.10), (4.11) and (4.12):

$$\begin{aligned} \text{Var}\left(V_T^{0,T,e}\right) &= \mathcal{V}(0, T, e) - e^2 \\ &= \mathcal{V}(0, T, \hat{e}) + e^{0,T}(e - \hat{e})^2 - e^2 \\ &= \frac{1}{v^{0,T}} + \frac{v^{0,T}}{v^{0,T} - (B_0^T)^2} \left(e - \frac{B_0^T}{v^{0,T}}\right)^2 - e^2 \\ &= \frac{(eB_0^T - 1)^2}{v^{0,T} - (B_0^T)^2} = \frac{(B_0^T)^2 \left(e - \frac{1}{B_0^T}\right)^2}{\text{Var}\left(\frac{Z^{opt*,0,T}}{B_T}\right)}. \end{aligned}$$

□

Lemma 4.3. *If the zero bond B^T exists in $\bar{\mathcal{M}}_{[0,T]}$ and $E[V] = \frac{1}{B_0^T}$ for all $V \in \mathcal{K}$, then $\text{Var}\left(\frac{Z^{opt*,0,T}}{B_T}\right) = 0$.*

Proof. Observe that $\text{Var}\left(\frac{Z^{opt*,0,T}}{B_T}\right) = v^{0,T} - (B_0^T)^2$ and that $\frac{1}{B_0^T} = E[V_T^{opt,0,T}] = \frac{B_0^T}{v^{0,T}}$ by (3.2). □

Definition 4.4. The *intertemporal price for risk* for maturity time T in the market $\bar{\mathcal{M}}_{[t,T]}$ is defined as

$$(4.15) \quad \beta_{[t,T]} := \frac{\sqrt{\text{Var}_t \left(\frac{B_t Z^{opt*,t,T}}{B_T} \right)}}{B_t^T}.$$

We have the following result:

Theorem 4.5. Assume the existence of the zero bond B^T in $\bar{\mathcal{M}}_{[0,T]}$.

Then the following inequality holds for all $H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^H = 1$:

$$(4.16) \quad \frac{1}{B_0^T} - \beta_{[0,T]} \sqrt{\text{Var}(V_T^H)} \leq E[V_T^H] \leq \frac{1}{B_0^T} + \beta_{[0,T]} \sqrt{\text{Var}(V_T^H)}.$$

In particular, $\beta_{[0,T]} = 0$ implies $E[V_T^H] = \frac{1}{B_0^T}$ for all $H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[0,T]})$, $V^{opt,0,T} = \frac{B^T}{B_0^T}$ and the so-called Return-to-Maturity Expectation Hypothesis for the zero bond price in $t = 0$ holds:

$$(4.17) \quad B_0^T = \frac{1}{E[B_T]}.$$

Furthermore, if $\beta_{[0,T]} \neq 0$, then

$$(4.18) \quad V^{0,T,e} = s(e)V^{opt,0,T} + (1 - s(e))\frac{B^T}{B_0^T},$$

where now $s(e) := -\frac{v^{0,T}B_0^T}{v^{0,T} - (B_0^T)^2} \left(e - \frac{1}{B_0^T} \right)$, and if

$$(4.19) \quad \left| E[V_T^H] - \frac{1}{B_0^T} \right| = \beta_{[0,T]} \sqrt{\text{Var}(V_T^H)},$$

for a $H \in \mathcal{SF}^{s,2}(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^H = 1$, then $V^H = V^{0,T,E}[V_T^H]$.

Proof. Inequality (4.16) follows from Lemma 4.2 and Lemma 4.3. If $\beta_{[0,T]} = 0$ we have $\frac{Z_T^{opt^*,0,T}}{B_T} = B_0^T$ a.s., since $\frac{Z_T^{opt^*,0,T}}{B_T}$ is almost surely deterministic and $E \left[V_T^{opt,0,T} \right] = E \left[\frac{Z_T^{opt^*,0,T}}{v^{0,T} B_T} \right] = \frac{B_0^T}{v^{0,T}} = \frac{1}{B_0^T}$. Hence $(B_0^T)^{-1} = E \left[\frac{Z_T^{opt^*,0,T}}{B_0^T} \right] = E[B_T]$. The remaining assertions follow from Proposition 4.1 and from the uniqueness of $V_T^{0,T,e}$, which implies the uniqueness of $V^{0,T,e}$. \square

Remark 4.6. The hedging numéraire has turned out to be the *market portfolio*, see Markowitz (1952, 1987). See Laurent and Pham (1999) and Leitner (2000) for explicit formulas for the hedging numéraire.

Corollary 4.7. *Assume the existence of the zero bond B^T in the market $\bar{\mathcal{M}}_{[0,T]}$. Then for all $H \in \mathcal{SF}^{sup,s,2}(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^H = 1$ and $E[V_T^H] \geq \frac{1}{B_0^T}$, the following inequality holds:*

$$(4.20) \quad E[V_T^H] \leq \frac{1}{B_0^T} + \beta_{[0,T]} \sqrt{\text{Var}(V_T^H)}.$$

Proof. V^H equals $V^{0,T,E[V_T^H]} + V^{H'}$ for a $H' \in \mathcal{SF}^{sup,s,2}(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^{H'} = 0$ and $E[V_T^{H'}] = 0$. Now calculate, using (4.18):

$$\begin{aligned} E \left[(V_T^H)^2 \right] &= E \left[\left(V_T^{0,T,E[V_T^H]} + V_T^{H'} \right)^2 \right] \\ &= E \left[\left(V_T^{0,T,E[V_T^H]} \right)^2 \right] + 2(1 - s(E[V_T^H])) E \left[\frac{1}{B_0^T} V_T^{H'} \right] \\ &\quad + 2s(E[V_T^H]) E \left[V_T^{opt,0,T} V_T^{H'} \right] + E \left[\left(V_T^{H'} \right)^2 \right] \\ &\geq E \left[\left(V_T^{0,T,E[V_T^H]} \right)^2 \right], \end{aligned}$$

where the last inequality follows from

$$E \left[V_T^{opt,0,T} V_T^{H'} \right] = E \left[\frac{V_T^{H'}}{v^{0,T} B_T} Z_T^{opt*,0,T} \right] \leq \frac{V_0^{H'}}{v^{0,T}} = 0$$

and $s(E[V_T^H]) \leq 0$ for $E[V_T^H] \geq \frac{1}{B_0^T}$. \square

Remark 4.8. The last result holds also for $H \in \mathcal{SF}(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^H = 1$, $E[V_T^H] \geq \frac{1}{B_0^T}$ and such that $\frac{V^H}{B^{[0,T]}} Z^{opt*,0,T}$ is a supermartingale. In particular, if $Z^{opt*,0,T} \in \mathcal{D}^2(\bar{\mathcal{M}}_{[0,T]})$, then the above result holds for all $H \in \mathcal{SF}^2(\bar{\mathcal{M}}_{[0,T]})$ with $V_0^H = 1$.

Remark 4.9. The results of this section hold in particular for the original one-step CAPM and its multi-period generalizations.

In the next section we will derive for a continuous price process \bar{S} similar results for the market $\bar{\mathcal{M}}_{[t,T]}$ using a stochastic duality approach.

5. THE CONDITIONAL PRICE FOR INTERTEMPORAL RISK

Let \bar{S} be continuous. Fix $0 \leq t \leq T < \infty$, and assume the zero bond B^T maturing at time T to be attainable in $\bar{\mathcal{M}}_{[t,T]}$, i.e. there exists a $H \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$ such that $V_T^H = 1$ almost surely. In this section we want to solve the optimization problem

$$(5.1) \quad \mathcal{V}(t, T, e, \mathcal{B}) := \text{ess inf}_{\substack{H \in \mathcal{B} \\ V_0^H = 1}} E_t \left[(V_T^H)^2 \right],$$

where $\mathcal{B} \in \{\mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]}), \mathcal{SF}_t^{sup,2}(\bar{\mathcal{M}}_{[t,T]}), \mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]})\}$, under the constraint

$$(5.2) \quad E_t[V_T^H] = e,$$

for an \mathcal{F}_t -measurable random variable e .

Since \mathcal{F}_0 was assumed to be trivial, we know that $Z_0^{opt*,0,T} = 1$. In the continuous case we also know that $Z^{opt*,0,T} > 0$, see GLP98. This allows to define $Z^{opt*,t,T} := \frac{Z_t^{opt*,t,T}}{Z_t^{opt*,t,T}} \in \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]})$. We have $V^{opt,0,T}$ with $V_0^{opt,0,T} = 1$ and $V_T^{opt,0,T} = \frac{Z_T^{opt*,0,T}}{v^{0,T} B_T} > 0$. Since $\frac{V^{opt,0,T}}{B^{[0,T]}} Z^{opt*,0,T}$

is a uniformly integrable martingale with $\frac{V_T^{opt,0,T}}{B_T} Z_T^{opt*,0,T} > 0$ we find $V^{opt,0,T} > 0$. This allows to define $V^{opt,t,T} := \frac{V_t^{opt,0,T}}{V_t^{opt*,0,T}} \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$. We then have

$$(5.3) \quad V_T^{opt,t,T} = \frac{Z_T^{opt*,t,T}}{v^{t,T} B_T},$$

where $v^{t,T} := \frac{V_t^{opt,0,T}}{Z_t^{opt*,t,T}} v^{0,T}$ is \mathcal{F}_t -measurable. Set

$$(5.4) \quad C_{[t,T]} := E_t \left[\left(\frac{B_t Z_T^{opt*,t,T}}{B_T} \right)^2 \right] = E_t \left[v^{t,T} V_T^{opt,t,T} \frac{B_t^2 Z_T^{opt*,t,T}}{B_T} \right] = B_t v^{t,T},$$

and note that

$$(5.5) \quad E_t \left[V_T^{opt,t,T} \right] = \frac{B_t^T}{C_{[t,T]}},$$

$$(5.6) \quad E_t \left[\left(V_T^{opt,t,T} \right)^2 \right] = \frac{1}{C_{[t,T]}},$$

and $\{\beta_{[t,T]} = 0\} = \{C_{[t,T]} = (B_t^T)^2\}$.

Lemma 5.1. *On $\{\beta_{[t,T]} = 0\}$, we have $\frac{B_t Z_T^{opt*,t,T}}{B_T} = B_t^T$ almost surely and for all $H \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$, resp. $H \in \mathcal{SF}_t^{sup,2}(\bar{\mathcal{M}}_{[t,T]})$, with $V_0^H = 1$ we have $E_t[V_T^H] = \frac{1}{B_t^T}$, resp. $E_t[V_T^H] \leq \frac{1}{B_t^T}$. Furthermore, on $\{\beta_{[t,T]} = 0\}$, $\beta_{[t',T]} = 0$ holds for all $t \leq t' \leq T$ and on $\{\beta_{[t',T]} = 0\}$ holds $B_{t'}^T = \frac{B_{t'}}{E_{t'}[B_T]}$ and $V^{opt,t',T} = \frac{B^T}{B_{t'}^T}$.*

Proof. Since $E_t \left[\frac{B_t Z_T^{opt*,t,T}}{B_T} \right] = B_t^T$, we find $\frac{B_t Z_T^{opt*,t,T}}{B_T}$ to be \mathcal{F}_t -measurable on $\{\beta_{[t,T]} = 0\}$, hence the first assertion holds. For $H \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$, resp. $H \in \mathcal{SF}_t^{sup,2}(\bar{\mathcal{M}}_{[t,T]})$, with $V_0^H = 1$ we find

$$E_t[V_T^H] = \frac{1}{B_t^T} E_t \left[V_T^H \frac{B_t Z_T^{opt*,t,T}}{B_T} \right] = \frac{1}{B_t^T},$$

resp. $E_t [V_T^H] \leq \frac{1}{B_t^T}$, on $\{\beta_{[t,T]} = 0\}$. By the definition of $Z^{opt*,t',T}$ we find $Z_T^{opt*,t',T}$ to be $\mathcal{F}_{t'}$ -measurable on $\{\beta_{[t,T]} = 0\}$, hence $\beta_{[t',T]} = 0$ there. Since $Z_T^{opt*,t,T} = \frac{B_T B_t^T}{B_t}$ on $\{\beta_{[t,T]} = 0\}$, we find $1 = E_t \left[\frac{B_T B_t^T}{B_t} \right] = B_t^T \frac{E_t[B_T]}{B_t}$ there. Now applying what we have proved so far to the case $t \leq t' \leq T$ we find the last assertion. \square

Proposition 5.2. *Let e be a \mathcal{F}_t -measurable random variable satisfying*

$e = (B_t^T)^{-1}$ on $\{\beta_{[t,T]} = 0\}$. Define $\Lambda_e := -\frac{C_{[t,T]} B_t^T}{C_{[t,T]} - (B_t^T)^2} \left(e - \frac{1}{B_t^T} \right)$ on $\{\beta_{[t,T]} \neq 0\}$, resp. $\Lambda_e := 0$ on $\{\beta_{[t,T]} = 0\}$. Then

$$(5.7) \quad V^{t,T,e} := \Lambda_e V^{opt,t,T} + (1 - \Lambda_e) \frac{B^T}{B_t^T}$$

is the unique solution of the constraint optimization problem (5.1) with respect to $\mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$, under the constraint e . On $\{\beta_{[t,T]} \neq 0\}$ we have

$$(5.8) \quad \mathcal{V}(t, T, e, \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})) = \frac{(B_t^T)^2}{C_{[t,T]} - (B_t^T)^2} \left(e - \frac{1}{B_t^T} \right)^2 + e^2,$$

resp. $\mathcal{V}(t, T, e, \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})) = (B_t^T)^{-2}$ on $\{\beta_{[t,T]} = 0\}$.

Proof. First, note that $V_0^{t,T,e} = 1$ and $E_t [V_T^{t,T,e}] = e$, hence $V^{t,T,e}$ is admissible for the constraint optimization problem (5.1). Define

$$(5.9) \quad F_{\Lambda_e}^{(t,T)}(x) := x^2 - 2 \frac{\Lambda_e}{v^{t,T}} \left(\frac{Z_T^{opt*,t,T}}{B_T} x - B_t^{-1} \right) - 2 \frac{1 - \Lambda_e}{B_t^T} (x - e).$$

$F_{\Lambda_e}^{(t,T)}$ is defined in such a way that for $H \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$, with $V_0^H = 1$ and $E_t [V_T^H] = e$, we have

$$(5.10) \quad E_t \left[F_{\Lambda_e}^{(t,T)}(V_T^H) \right] = E_t \left[(V_T^H)^2 \right].$$

Furthermore, since

$$\frac{dF_{\Lambda_e}^{(t,T)}}{dx}(x) = 2x - 2\frac{\Lambda_e}{v^{t,T}} \frac{Z_T^{opt*,t,T}}{B_T} - 2\frac{1-\Lambda_e}{B_t^T},$$

and by (5.3)

$$(5.11) \quad \frac{dF_{\Lambda_e}^{(t,T)}}{dx}(x) = 0 \Leftrightarrow x = \frac{\Lambda_e}{v^{t,T}} \frac{Z_T^{opt*,t,T}}{B_T} + \frac{1-\Lambda_e}{B_t^T} = V_T^{t,T,e},$$

and $\frac{d^2 F_{\Lambda_e}^{(t,T)}}{dx^2} > 0$, we find

$$(5.12) \quad F_{\Lambda_e}^{(t,T)}(V_T^{t,T,e}) = \inf_{x \in \mathbb{R}} F_{\Lambda_e}^{(t,T)}(x).$$

Therefore

$$\begin{aligned} E_t \left[\left(V_T^{t,T,e} \right)^2 \right] &= E_t \left[F_{\Lambda_e}^{(t,T)}(V_T^{t,T,e}) \right] \\ &\leq E_t \left[F_{\Lambda_e}^{(t,T)}(V_T^H) \right] = E_t \left[(V_T^H)^2 \right] \end{aligned}$$

Now calculate on $\{\beta_{[t,T]} \neq 0\}$, using $e = \frac{\Lambda_e B_t^T}{C_{[t,T]}} + \frac{1-\Lambda_e}{B_t^T}$ and $e^2 = \frac{\Lambda_e B_t^T}{C_{[t,T]}} e + \frac{1-\Lambda_e}{B_t^T} e$:

$$\begin{aligned} \mathcal{V}(t, T, e, \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})) &= E_t \left[\left(V_T^{t,T,e} \right)^2 \right] = \\ &= E_t \left[\left(\Lambda_e V_T^{opt,t,T} + \frac{1-\Lambda_e}{B_t^T} \right) V_T^{t,T,e} \right] \\ &= \Lambda_e E_t \left[V_T^{opt,t,T} V_T^{t,T,e} \right] + \frac{1-\Lambda_e}{B_t^T} e \\ &= \Lambda_e E_t \left[\left(V_T^{opt,t,T} \right)^2 \right] + e^2 - \frac{\Lambda_e B_t^T}{C_{[t,T]}} e \\ &= -\frac{\Lambda_e B_t^T}{C_{[t,T]}} \left(e - \frac{1}{B_t^T} \right) + e^2 = \frac{(B_t^T)^2}{C_{[t,T]} - (B_t^T)^2} \left(e - \frac{1}{B_t^T} \right)^2 + e^2. \end{aligned}$$

On $\{\beta_{[t,T]} = 0\}$ we have $\mathcal{V}(t, T, e, \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})) = (B_t^T)^{-2}$ by Lemma

5.1. Uniqueness of $V^{t,T,e}$ follows from strict convexity. \square

Theorem 5.3. *Assume the existence of the zero bond B^T in $\bar{\mathcal{M}}_{[t,T]}$.*

Then the following inequality holds for all $H \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})$ with $V_0^H =$

1:

$$(5.13) \quad \frac{1}{B_t^T} - \beta_{[t,T]} \sqrt{\text{Var}_t(V_T^H)} \leq E_t[V_T^H] \leq \frac{1}{B_t^T} + \beta_{[t,T]} \sqrt{\text{Var}_t(V_T^H)}.$$

Furthermore,

$$(5.14) \quad \left| E_t[V_T^H] - \frac{1}{B_t^T} \right| = \beta_{[t,T]} \sqrt{\text{Var}_t(V_T^H)},$$

holds if and only if $V^H = V^{t,T,E_t[V_T^H]}$ on $\{\beta_{[t,T]} \neq 0\}$. On $\{\beta_{[t,T]} = 0\}$

the Return-to-Maturity Expectation Hypothesis holds:

$$(5.15) \quad B_{tV}^T = \frac{B_{tV}}{E[B_T | \mathcal{F}_{tV}]}$$

Proof. By Proposition 5.2 we find on $\{\beta_{[t,T]} \neq 0\}$

$$(5.16) \quad \beta_{[t,T]} = \frac{\left| e - \frac{1}{B_t^T} \right|}{\sqrt{\text{Var}_t(V_T^{t,T,e})}} = \max_{\substack{H \in \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]}) \\ V_0^H = 1, E[V_T^H] = e}} \frac{\left| e - \frac{1}{B_t^T} \right|}{\sqrt{\text{Var}_t(V_T^H)}}.$$

This and Lemma 5.1 imply the first assertion. The second assertion

follows from the uniqueness of $V^{t,T,e}$ on $\{\beta_{[t,T]} \neq 0\}$. The last assertion

follows again from Lemma 5.1. \square

Corollary 5.4. *Assume the existence of the zero bond B^T in $\bar{\mathcal{M}}_{[t,T]}$.*

Then the following inequality holds for all $H \in \mathcal{SF}_t^{\text{sup},2}(\bar{\mathcal{M}}_{[t,T]})$ with

$V_0^H = 1$ and $E_t[V_T^H] \geq \frac{1}{B_t^T}$:

$$(5.17) \quad E_t[V_T^H] \leq \frac{1}{B_t^T} + \beta_{[t,T]} \sqrt{\text{Var}_t(V_T^H)}.$$

Proof. For $H \in \mathcal{SF}_t^{sup,2}(\bar{\mathcal{M}}_{[t,T]})$, with $V_0^H = 1$ and $E_t[V_T^H] = e \geq \frac{1}{B_t^T}$,

we have, see (5.9),

$$(5.18) \quad E_t \left[F_{\Lambda_e}^{(t,T)}(V_T^H) \right] \leq E_t \left[(V_T^H)^2 \right],$$

since $\Lambda_e \leq 0$ for $e \geq \frac{1}{B_t^T}$, hence

$$(5.19) \quad \mathcal{V}(t, T, e, \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})) \leq E_t \left[(V_T^H)^2 \right],$$

which implies the assertion. \square

In the special case of a deterministic B , or working with the discounted market $\bar{\mathcal{M}}_{[t,T]}^* := (\Omega_{[t,T]}, (\frac{S}{B}, 1)^{[t,T]})$, where zero bonds trivially exist for all maturity times, the intertemporal price for risk, denoted as $\beta_{[t,T]}^*$ in the market $\bar{\mathcal{M}}_{[t,T]}^*$, is related to results by DMSSS97, especially Theorem B, where for $(\frac{S}{B})^{[t,T]} \in \mathcal{S}_{loc}^2(\Omega_{[t,T]})$, the closedness of $\mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]}^*)$ is shown to be equivalent to the (non-discounted) variance optimal martingale measure in $\bar{\mathcal{M}}_{[t,T]}^*$, denoted as $Z^{opt,t,T}$, satisfying the so-called reverse Hölder inequality:

$$(5.20) \quad E_s \left[\left(\frac{Z_T^{opt,t,T}}{Z_s^{opt,t,T}} \right)^2 \right] \leq K, \quad \forall t \leq s \leq T,$$

for a constant K . This condition is equivalent to

$$(5.21) \quad \beta_{[s,T]}^* \leq \sqrt{K-1} \quad \forall t \leq s \leq T,$$

since $Z^{opt,s,T} = \frac{Z_{s \vee \cdot}^{opt,t,T}}{Z_s^{opt,t,T}}$ for all $t \leq s \leq T$ and

$$\beta_{[s,T]}^* = \sqrt{E_s \left[\left(\frac{Z_T^{opt,t,T}}{Z_s^{opt,t,T}} \right)^2 \right] - 1}.$$

For an \mathcal{F}_t -measurable random variable $e \geq 1$, such that $e = 1$ on $\{\beta_{[t,T]}^* = 0\}$, denote the solution for the constraint optimization problem (4.1) in the discounted market $\bar{\mathcal{M}}_{[t,T]}^*$ by $V^{*,t,T,e}$. For $V := V_{\cdot \wedge T}^{*,t,T,e}$, which can be seen as the value process of a self-financing hedging strategy in $\bar{\mathcal{M}}_{[t,T']}^*$ for $T \leq T'$, we have $E_t[V_{T'}] = 1 + \beta_{[t,T]}^* \sqrt{\text{Var}_t(V_{T'})}$. Therefore

$$(5.22) \quad \beta_{[t,T]}^* \leq \beta_{[t,T']}^*, \quad \forall T \leq T'.$$

We summarize these observations:

Theorem 5.5. For $T > 0$ let $\left(\frac{S}{B}\right)^{[0,T]} \in \mathcal{S}_{loc}^2(\Omega_{[0,T]})$. We then have

equivalence between

1. $\mathcal{G}^2(\bar{\mathcal{M}}_{[0,T]}^*)$ is closed.
2. $\{\beta_{[t,T]}^* | 0 \leq t \leq T\}$ is uniformly bounded.
3. $\{\beta_{[t,T']}^* | 0 \leq t \leq T' \leq T\}$ is uniformly bounded.
4. $\mathcal{G}^2(\bar{\mathcal{M}}_{[t,T']}^*)$ is closed for all $0 \leq t \leq T' \leq T$.

Corollary 5.6. If $\mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]})$ is closed and if B^T is attainable in $\bar{\mathcal{M}}_{[t,T]}$ with a self-financing hedging strategy in $\mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]})$, then

$$(5.23) \quad \mathcal{V}(t, T, e, \mathcal{SF}_t^2(\bar{\mathcal{M}}_{[t,T]})) = \mathcal{V}(t, T, e, \mathcal{G}^2(\bar{\mathcal{M}}_{[t,T]})).$$

6. APPLICATION

In an incomplete market with zero bond, one way to price non-attainable claims is to price them with respect to an equivalent martingale measure that is in some sense optimal, e.g. minimal, variance-optimal, L^q -optimal, entropy minimal. If the discounted variance-optimal measure is an equivalent probability measure, it has the special property that the intertemporal price for risk $\beta_{[t,T]}$ for maturity time T in the market $\bar{\mathcal{M}}_{[t,T]}$ remains unchanged if new securities priced according to it are introduced to the market: For a non-attainable square integrable \mathcal{F}_T -measurable contingent claim \bar{X} we can define the price process $X_s := \frac{B_s^{[t,T]} E_s \left[\frac{\bar{X}}{B_T} Z_T^{opt^*,t,T} \right]}{Z_s^{opt^*,t,T}}$. $\frac{X}{B^{[t,T]}}$ is a uniformly integrable martingale with respect to the discounted variance optimal martingale measure of the market $\bar{\mathcal{M}}_{[t,T]}$ defined by $Z_T^{opt^*,t,T}$. Therefore, for the extended market $\bar{\mathcal{M}}_{[t,T]}^X := (\Omega_{[t,T]}, (\bar{S}, X)^{[t,T]})$ with intertemporal price for risk denoted as $\beta_{[t,T]}^X$, we have $Z^{opt^*,t,T} \in \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]}^X) \subseteq \mathcal{D}_t^2(\bar{\mathcal{M}}_{[t,T]})$, implying $Z^{opt^*,t,T}$ to be the discounted variance optimal martingale

measure for the extended market $\bar{\mathcal{M}}_{[t,T]}^X$ and $\beta_{[t,T]}^X = \beta_{[t,T]}$. X_t is also known to be the initial price of the mean-variance optimal self-financing hedging strategy approximating \bar{X} .

In general it is not easy to calculate any of the quantities $\beta_{[t,T]}$, $C_{[t,T]}$ and B_t^T , which are related in the following way:

$$(6.1) \quad \beta_{[t,T]} = \frac{\sqrt{C_{[t,T]} - (B_t^T)^2}}{B_t^T}.$$

(This equation follows immediately from the definition of $\beta_{[t,T]}$.) In Leitner (2000), an example is given where $C_{[t,T]}$ can be calculated explicitly. In a markovian setting a PDE is derived, from which $C_{[t,T]}$ can be calculated.

Estimating the function $t \mapsto C_{t_0, t_0+t}$, $t_0 \leq t_0+t \leq t_1$, from historical data and calculating $\beta_{[t_0, t_0+t]}$ via equation (6.1) using historical zero bond prices, one can try to find a model for the quantities C_{t_1, t_1+t} and β_{t_1, t_1+t} . Solving (6.1) for $B_{t_1}^{t_1+t}$ we find a model for the zero bond prices, which can be compared to observed prices. Alternatively, one can estimate $\beta_{[t_1, t_1+t]}$, $t > 0$ from observed zero bond prices and a model for C_{t_1, t_1+t} and look for interesting patterns in the graph of $t \mapsto \beta_{[t_1, t_1+t]}$.

7. CONCLUSIONS

We have shown that the term-structure of interest rates and the term structure of intertemporal prices for risk are closely related.

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