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NEYMAN-PEARSON HEDGING AND DYNAMIC MEASURES OF RISK

M. KOHLMANN

ABSTRACT. In both complete and incomplete markets we consider the problem of fulfilling a financial obligation ξ_c as well as possible at time T if the initial capital is not sufficient to hedge ξ_c. This introduces a new risk into the market and our main aim is to minimize this shortfall risk by making use of results from bsde theory.

1. INTRODUCTION

The problem of hedging a claim ξ at time $t = T$ attracted new interest in the recent literature under the point of view that there is no sufficient initial capital x to perfectly hedge the liability ξ. In a complete market M the price of the claim is completely described (- under appropriate technical assumptions -) as

$$E^*(\xi|F_t), 0 \leq t \leq t$$

where E^* is the risk neutral measure appropriately discounted, or as the solution of a backward stochastic differential equation

$$dp_t = (rp_t + \theta_t q_t)dt + q_t dw_t, 0 \leq t \leq T$$
$$p_T = \xi$$

where r is the interest rate and θ is the risk premium process. If the investors' initial capital x is strictly less than p_0, there is no chance to perfectly hedge the liability and so there is additional risk in the market which must be measured.

Artzner, Delbaen, Eber and Heath [1] use risk measures in a static setting satisfying certain coherence properties and Cvitanic and Karatzas [3], [4] introduce a dynamic version of this measure

$$\lambda(x, \zeta) = \inf_{\pi \in A(x)} E(\tfrac{\xi - x^{x,\pi}(T)}{P_0(T)})^+$$

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where $A(x)$ is a set of admissible portfolios, $P_0(t)$ is the riskless instrument in the market and E is the expectation with respect to the real world measure. $x^{x,\pi}(t)$ is the agent's wealth at time t when starting with initial capital x and when strategy π is used. $\lambda(x,\zeta)$ describes the expected size of shortfall, if $x < p_0$. It is one of the many possibilities described in the recent literatur to measure and quantify the genuine risk ξ. In generality such measures have been studied by Artzner, Delbaen, Eber and Heath [1]. A different approach is described by Föllmer and Leukert [11], [12], where the agent tries to maximize the probability of a perfect hedge

$$\sup_{\pi \in A(x)} P(x^{x,\pi}(T) \geq \xi).$$

This approach makes use of results from test theory in statistics and thus may be seen as the dynamic version of the familiar "value at risk" concept which is described e.g. by Heath, Kulldorf, and Cvitanic and Karatzas [15], [20], [3].

This "value at risk" concept does not take into account the size of shortfall. So the dynamic measure $\lambda(x,\xi)$ must be preferred to the later approach. Föllmer and Leukert [12] study a dynamic measure of the form

$$\lambda(x,\xi) = \inf_{\pi \in A(x)} E\left[l\left(\frac{\xi - x^{x,\pi}}{P_0(T)} \right)^+ \right]$$

where l is a concave function and they study the relation to the "statistical" approach.

Kohlmann and Zhou [18] choose a mean variance hedging approach to describe the agent's behaviour when the initial capital is different from the hedging price of the obligation ξ. This approach is also treated in [14], [5] and [6]. The main disadvantage of this approach is the fact that no difference is made between overshooting and shortfall. Nevertheless it leads to useful results in special situations (see the explanations in section 3.1).

We shall first use these results to derive the results on the dynamic measure $\lambda(x,\xi)$ in a complete market. As we shall make extensive use of recent results on backward stochastic differential equations (bsde) this approach is completely different from the methods in the literature cited above. For results on bsdes the reader is referred to [2], [22], [23], [24], [27], and [29]. The statistical tools are found in [28], and the results used from martingale theory in [21], [9].

The next section will show how to extend the results to incomplete markets. Again the methods to derive the results base on techniques from the theory of bsdes.

In the last section we extend the ideas to the problem of measuring the shortfall risk in the American contingent claim case and we suggest a completely different method for treating the problem which only relies on FBSDE techniques.

2. The market tools

This section describes the market M which we are going to consider first. It also gives the mathematical tools used in the later sections as the basic results on hedging, on backward stochastic differential equations etc.

2.1. **The Market.** We consider the usual B-S-market described by one bond P_0 and d stocks $(P_1(t), ...P_d(t))$

$$dP_0(t) = r(t)P_0(0)dt, P_0(0) = 1$$
$$dP_i(t) = P_i(t)b_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dw^{j(t)}, P_i(0) = p_i > 0$$
$$0 < t \leq T$$

on a probability space (Ω, F, P) on which the d-dimensional Brownian motion (w_t) lives.

The natural filtration is the P-augmentation of the filtration generated by $(w^1, .., w^d)$, say $F_t = F_t^w$. For simplicity we assume that all coefficients $r(t), b_i(x), \sigma_{i,j}(t)$ are deterministic, which will us allow to directly apply the results of Kohlmann-Zhou [18]; the generalization to progressively measurable coefficients will not encounter systematic difficulties and will be described in a forthcoming paper, where we consider a more general framework to describe measures of risk. Furthermore $\sigma = (\sigma_{i,j})$ is assumed to be invertible and - again for simplicity - all coefficients $r, b, \sigma_{i,j}$, and those of σ^{-1} are assumed to be bounded.

The above assumptions allow us to define the bounded risk premium process

$$\theta(t) = \sigma^{-1}(t)(b(t) - r(t) \cdot 1), 0 \leq t \leq T$$

the existence of which assures that the market is arbitrage free (see e.g. [9], [16]). The Girsanov functional associated with $\theta(t)$, namely the solution of

$$dz_{0t} = -\theta z_{0t} dw_t, z_{00} = 1$$

is a P-martingale and defines the risk neutral measure by

$$d\widetilde{P} = z_{0T} dP$$

under which the discounted stocks become martingales. From Girsanov's theorem we then have a Brownian motion under $\widetilde{P}$ namely

$$\widetilde{w}_t = w_t + \tfrac{1}{2} \int_0^t \theta_s ds$$

The expectations with respect to P and $\widetilde{P}$ will be denoted by E and $\widetilde{E}$, respectively.

With these tools we find ourselves in the standard arbitrage free, complete market. In this market the wealth process of an agent who starts with an initial capital x and follows a self-financing portfolio strategy $\pi(t) = (\pi_1(t), ..., \pi_d(t))$ where $\pi_i(t)$ is the amount invested in the $i-th$ stock, is given by

$$\begin{gathered} dx_t = (r(t)x(t) + \theta_j(t)u_j(t))dt + u_j(t)dw_t^j. \\ x_0 = x \end{gathered}$$

where $u = \sigma\pi$. The above wealth equation under $\widetilde{P}$ is obviously given by

$$dx_t = r(t)x(t)dt + u_j d\widetilde{w}_t^j, x_0 = x.$$

A portfolio π (or equivalently a control u) is an R^d-valued process on $[0,T] \times \Omega$ which we assume to be progressively measurable and almost surely twice integrable.

We define the deflator process

$$dz_{0t}^r = -r_t z_{0t}^r dt - \theta z_{0t}^r dw_t, z_{00} = 1$$

and the associated deflated measure is defined by

$$d\widetilde{P}^r = z_{0t}^r dP.$$

Note that the following relations hold

$$d\widetilde{P}^r = H_{0T}^r z_{0T} dP = H_{0T}^r d\widetilde{P}$$

where

$$dH_{0t}^r = -r_t H_{0t}^r dt, H_{00} = 1$$

and where $\widetilde{P}$ is the risk neutral measure.

As we shall mainly work with $\widetilde{P}^r$ instead of $\widetilde{P}$ we will write $\widetilde{P}$ for the deflated risk neutral measure and z_{0t} for the corresponding density as long as we expect no confusion. P will then be referred to as the 'measure in reality' or 'physical measure' and $\widetilde{P}$ will be called the virtual measure. The reader who might find this confusing should at first reading just let $r = 0$.

Finally, a contingent claim ξ is an $L^2(F_T, P)-$random variable, which may be thought of as a bounded Borel function of $P(T) = (P_1(T), ..., P_d(T))$, say

$$\xi = g(P_T)$$

and the best known example is the European option

$$\xi = g(P_T) = (P_T - K)^+,$$

where K is the exercise price of the option.

2.2. **Pricing a contingent claim and mean variance hedging.** As pointed out in Kohlmann [17] we consider the formal adjoint for the following trivial control problem

$$dz_{0t} = -rz_{0t}dt - \theta z_{0t}dw_t, z_{00} = 1$$
$$E(z_{0T}\xi) = \min!$$

which is given by the backward stochastic differential equation

$$dp_t = (r_t p_t + \theta q_t)dt + q_t dw$$
$$p_T = \xi.$$

Meanwhile it is standard that the solution of the above bsde

$$(p, q) = (p^\xi, q^\xi)$$

gives the price of the option ξ, namely

$$p_t = \widetilde{E}(\xi|F_t) = E(z_{tT}\xi/F_t)$$

and q_t is the hedging portfolio process in the sense that

$$\pi(t) = \sigma^{-1}q(t).$$

For future reference we summarize these results in

Theorem 2.1. *Under all the above conditions the adapted solution of the above bsde (p, q) exists. The process (p_t) describes the fair price of the contingent claim and (q_t) is the portfolio process for optimally hedging the claim.*

Remark 2.2. We obviously restrict our considerations first to the simplest case possible. As noted above already we might allow for coefficients depending on ω in a progressively measurable way. Also we could consider constraints on the portfolio process, like cone constraints for instance. There is a vast literature on these models, the generalization is straightforward.

In Kohlmann-Zhou [18] we consider the following mean variance hedging problem

$$dx_t = (r_t x_t + \theta_t u_t)dt + u_t dw_t, x_0 = x \in R^d$$

with cost criterion

$$J(x, u) = \tfrac{1}{2}E(x_T - \xi)^2 = \min!$$

From a general LQ-control problem we derive the following result

Theorem 2.3. *(i): The optimal portfolio $(u = \sigma\pi)$ of the above hedging problem is*

$$\pi^* = \theta(t)(x^*(t) - p_t) + \sigma_t^{-1}q_t$$

where

$$(p, q) \in L^2([0, T], R^n, (F_t)) \times L^2([0, T], R^n, (F_t))$$

is the unique F_t-adapted solution of

$$dp_t = [r(t)\, p_t + \theta_t q_t]\, dt - q dw$$
$$p_T = \xi$$

(ii) the optimal value of the hedging problem is

$$\tfrac{1}{2}\Gamma(T) = \tfrac{1}{2} P(0) (x - p(0))^2$$

where P solves the stochastic Riccati equation

$$\dot{P}(t) + 2r(t)P(t) - \Sigma\theta_j^2(t) P(t) = 0$$
$$P(T) = 1, P(t) > 0, t \in [0, T]$$

and the error

$$\Gamma(t) = E(x_t^* - p_t)^2$$

satisfies

$$\dot{\Gamma} - 2(r - \theta^2) - \theta^2\Gamma = 0$$
$$\Gamma(0) = (x - p(0))^2$$

(iii) The smallest initial endowment to reach the contingent claim ξ is

$$x = p(0)$$

and in this case the wealth process $(x_t^{x,\pi*}) = (x_t^*)$ is equal to (p_t). So the fair price of the contingent claim (- and so the solution of the bsde (p_t) -) is the wealth process for the mean variance hedging problem with starting point $x = p(0)$, and (q_t) is the corresponding portfolio multiplied by σ.

Note that the above result may be interpreted in the following way: If the initial capital is equal to the fair price of the contingent claim at time $t = 0$ then the wealth process is just the price of the contingent claim. Let us assume that the agent has a higher (respectively, a lower) capital then the 'rest' is invested according to a Merton type strategy.

2.3. **Superhedging and the upper price in an incomplete market.** When we consider incomplete markets the target ξ might not be reachable, the liability ξ cannot be hedged in the sense that there might not be a starting point x and a portfolio π such that $x_T^{x,\pi} = \xi$. This situation typically arises when the number of tradeable securities is smaller than the number of random sources in the market. A most successful approach to solving this problem is the Föllmer et al. hedging or the mean variance hedging approach [14]. The research in this field gives a deep insight into the structure of hedging problems, though the main drawback of this theory is the fact that mean variance hedging makes no difference between a shortfall from the unreachable obligation and an overshooting: The influence of these two failures of completing a liability are ruled out by just considering the variance.

A completely different approach was taken by Karatzas, Shreve, Cvitanic, El Karoui and Quenez [3], [4] ,[8] ,[16]. The idea consists in defining a self-financing superstrategy (π, c) where π is a portfolio process and the increasing right continuous process (c_t) with $c_0 = 0$ may be interpreted as a consumption process. The upper price of a contingent claim is then given by the generalized bsde

$$dp_t = (rp_t + \theta q_t)\, dt - dc_t + q_t dw_t$$
$$p_T = \xi.$$

It is straightforward that the idea of introducing the above generalized bsde has its counterpart in the martingale approach. Karatzas and Shreve [16] make use of the classical Doob Meyer decomposition to derive the existence of the cumulative consumption process. A similar approach was taken by El Karoui and Quenez [8]. Generalizations are found in Kramkow [10], and Föllmer-Kramkow [19], where a "Doob-Meyer decomposition" with an optional process is derived. Yet another approach is possible by considering $g - martingales$ [27].

At this point it makes sense to formalize the bsde techniques for future use. The definitions and properties are taken from [24]. Many results below may be cited in a more general way, we restrict the assumptions to our needs.

Given (Ω, F, P) and an R^n-valued Brownian motion so we consider

(i) $(F_t)_{t\in[0,T]}$ the augmented filtration generated by $(w_t)_{t\in[0,T]}$ and it is assumed that the filtration satisfies the usual properties. $\mathfrak{P}$ denotes the predictable σ-field.

(ii) $L^2\left(T; R^d\right)$, the space of all F_T-measurable random variables

$$x\colon \Omega \longrightarrow R^d \text{ with } E\left(|x|^2\right) < \infty.$$

(iii) $H^2\left(T; R^d\right)$ the space of predictable processes

$$\phi : \Omega \times [0,T] \longrightarrow R^d \text{ with } E_0^T \int \left(\phi_t\right)^2 dt < \infty.$$

(iv) $H^1\left(T; R^d\right)$ the space of predictable processes

$$\phi : \Omega \times [0,T] \longrightarrow R^d \text{ with } E\sqrt{\int_0^T |\phi_t|^2 dt} < \infty$$

v) $H^2\left(T, R^d\right), \beta > 0$ denotes the space $H^2\left(T; R^d\right)$ endowed with the norm $|.|$ defined by $|.|^2 = E_0^T \int e^t \phi_t^2 dt$.

Definition 2.4. Consider the bsde

$$dp_t = -f\left(t, p_t, q_t\right) dt + q_t dw_t,\ p_T = \xi$$

as an anologue of

$$p_t = \xi + \int_t^T f\left(t, p_t, q_t\right) dt - \int_t^T q_t^T dw_t$$

where ^{T} here denotes the transpose and where
(i) ξ is an F_T-measurable random variable
(ii) the "generator" f,

$$f : \Omega \times R^* \times R^d \times R^{n\times d} \longrightarrow R^d$$

is $\mathfrak{P} \otimes B^d \otimes B^{n\times d}$-measurable.
We call (p_t,q_t), such that
(i) (p_t) is an adapted continuous R^d-valued process
(ii) (q_t) is an $R^{n\times d}$valued predictable process with $\int_\delta^T |q_s|^2 ds < \infty, P-a.s.$ a solution of the above bsde.

The next theorem gives a fairly general existence result [26]

Theorem 2.5. *Let $\xi \in L^2\left(T; R^d\right), f\left(.,0,0\right) \in H^2\left(T; R^d\right)$ such that there exists $C > 0$ with*
$\left|f\left(\omega,t,p_1,q_1\right) - f(\omega,t,p_2,q_2)\right| \leq C(|p_1-p_2| + |q_1-q_2|), P-a.s.$ for all p_1,p_2,q_1,q_2. (In [8] (f,ξ) is then called a pair of standard parameters for the bsde).
Then there exists a unique pair $(p,q) \in H^2\left(T; R^d\right) \times H^2\left(T; R^{n\times d}\right)$ which solves the above bsde.

We shall make use of a comparison theorem which is stated in

Theorem 2.6. *Let $\left(f^1,\xi^1\right), \left(f^2,\xi^2\right)$ be standard parameters of bsdes and let (p^1,q^1) and (p^2,q^2) be the associated square integrable solutions. Let*
(i) $\xi^1 \geq \xi^2, P-a.s.$
(ii) $f^1(t,p_t^1,q_t^1) - f^2(t,p_t^2,q_t^2) \geq 0, dP\otimes dt - a.s.$
Then for any t
$p_t^1 \geq p_t^2$
almost surely. If $p_t^1 = p_t^2$ on $A \in F_t$ then
(i) $p_t^1 = p_t^2$ on $[t,T]\times A, a.s.$
(ii) $\xi^1 = \xi^2, a.s.$ on A
(iii) $f^1\left(s,p_s^1,q_s^1\right) = f^2\left(s,p_s^2,q_s^2\right)$on $A\times[t,T], dP\otimes ds - a.s.$

The comparison theorem implies

Corollary 2.7. *If $\xi \geq 0$ a.s. and $f\left(t,0,0\right) \geq 0$ $dP\otimes dt -$ a.s. then $p_t \geq 0$ $P-a.s.$ If $p_t = 0$ on a set $A \in F_t$,then $p_s = 0, f(s,0,0) = 0$ on $[t,T]\times A$ $dP\otimes ds - a.s.$ and $\xi = 0$ a.s. on A.*

Remark 2.8. We shall mainly use the above theorems for the linear case, that is for the case
$-f\left(t,p,q\right) = r_t p_t + \theta_t q_t + \phi_t$
where r_t, θ_t are bounded R, resp. R^n-valued predictable process $\phi \in H^2\left(T, R^d\right)$. It is straightforward from the above theorem that in this case the bsde has a unique solution and (p_t) is given by

$$p_t = E\left[z_{0t}^{-1} z_{0T}\xi + \int_t^T z_{0t}^{-1} z_{0s}\phi_s ds|F_t\right] \; P-a.s.$$

Definition 2.9. A supersolution of a bsde with standard parameters (f,ξ) is a triple of processes (p,q,C) with

$$dp_t = -f\left(t,p_t,q_t\right)dt + q_t^* dw_t - dC_t, p_T = \xi$$

where

- ξ is F_T-measurable
- (p_t) is a cadlag process.
- (q_t) is a predictable process with $\int_0^T q_s^2 ds < \infty \; P-a.s.$
- (C_t) is an increasing, adapted, right continuous process with $C_0 = 0$
- $p_t \geq 0 \; P-a.s.$

Theorem 2.10. *(i) Let (p^1,q^1) and (p^2,q^2) be two solutions of bsdes with standard parameters satisfying the assumption of the comparison theorem. Then there exists a process C such that (p^1,q^1,C) is a supersolution of the bsde with parameter (f^2,ξ^2).*

(ii) Let $(p^1,q^1,C^1)$and (p^2,q^2,C^2)

be two continuous supersolutions of bsdes with parameters

$$(f^1,\xi^1) \text{ and } (f^2,\xi^2).$$

Then there exists a pair (q^*,C^*)such that $(p^* = p^1 \wedge p^2, q^*, C^*)$ is a supersolution of the bsde with terminal condition $\xi^* = \xi_1 \wedge \xi_2$ and generator

$$f^t\left(t,p,q\right) = 1_{\{p_t^1 \leq p_t^2\}} f^1\left(t,p,q\right) + 1_{\{p_t^2 < p_t^1\}} f^2\left(t,p,q\right).$$

$(q^*)$is explicitly given by

$$q_t^* = 1_{\{p_t^1 \leq p_t^2\}} q_t^1 + 1_{[p_t^2 < p_t^1\}} q_t^2$$

and C^* is given by

$$dC_t^* = 1_{\{p_t^1 \leq p_t^2\}} dC_t^1 + 1_{\{p_t^2 < p_t^1\}} dC_t^2 + dL_t$$

where (L_t) is a local time, i.e. an increasing continuous process with support included in $\{t|p_t^1 = p_t^2\}$.

Finally we introduce the notion of a reflected bsde. The idea of this definition is to define a process (p_t) which is a supersolution of a bsde such that (p_t) stays above a pregiven process (S_t) with $S_T = \xi$ or more general $S_T \leq \xi$.

Definition 2.11. Let f be an R-valued standard driver and let (S_t) be a continuous process on $(0,T)$, adapted, real valued and L^2-bounded with

$$\lim_{t \to T} S_t \leq \xi \; P-a.s.$$

The triple (p_t, q_t, K_t) of processes taking values in R, R^n, R_+ respectively, and satisfying

Definition 2.12. (i) $(q_t) \in H^{2,d}, (p_t) \in S^2, K_T \in L^{2,d}$
(ii) $p_t = \xi + \int_t^T f(s.p_s.q_s)\, d_s + K_T - K_t - \int_t^T q_s^* dw_s$
(iii) $p_t \geq S_t$
(iv) K is continuous and increasing with

$$\int_0^T (p_t - S_t) dK_t = 0$$

is said to be a solution of the reflected bsde, if

$$dp_t = -f(s, p_t, q_t)\, dt - dK_t + q_t dw_t$$
$$P_T = \xi, p_t \geq S_t \ \ P.a.s.$$

Now let for (f, S_t, ξ) the properties in Definition 2.3.9 be satisfied. (f, S_t, ξ) is then called a set of standard data. We may cite the following

Theorem 2.13. *(i) Let (f, S_t, ξ) be a set of standard data. Then the rbsde has a unique square integrable solution.*

(ii) Let $(f^1, S_t^1, \xi^1), (f^2, S_t^2, \xi^2)$ be standard datas and let (p^1, q^1, K^1) and (p^2, q^2, K^2)be the solutions of the rbsdes. If

$$S_t^1 \leq S_t^2, \xi^1 \leq \xi^2 \ P - a.s.$$

and

$$f^1(t, p_t^1, q_t^1) \leq f^2(t, p_t^2, q_t^2)\, dP \times dt \ \ - a.s.$$

Then

$$p_t^1 \leq p_t^2 \ P - a.s.$$

3. Measuring risk and Neyman-Pearson hedging

Let us assume that the agent has an initial capital x_0. At time $t = T$ his financial obligations are described by the contingent claim ξ_c. If the initial capital is greater than the price of ξ_c at time $t = 0$, i.e. $\mathrm{p}_c(0)$, then the agent is able to fulfill his obligations. A problem and an additional risk arises when

$$x_0 < p_c(0).$$

Cvitanic and Karatzas [3], [4] consider the situation where an observer of the situation at time $t = 0$ has information about ξ_c and about the agent's aim to hedge ξ_c. The observer requires that the agents wealth stays above the price $p_a(t)$ of a contingent claim ξ_a, for which then of course $x_0 \geq p_a(0)$, and $\xi_c \geq \xi_a$. For the economical interpretation see [3]. Instead the observer could also require that the agent should fulfill his obligation as well as he can under the additional condition that his present wealth at time t should be near the estimated value of a portion $k\xi_c, k \in (0, 1)$ of the contingent claim.

Let $A(x)$be the family of admissible strategies π. We take the least expected loss with respect to $\pi \in A(x)$ as measure of risk:

$$R(x) = R(x, \xi_c) = \inf_{\pi \in A(x)} E\left[H_{0T}^r (\xi_c - x_T^{x,\pi})^+\right]$$

Remark 3.1. Artzner et al. [1] propose certain desirable properties of a risk measure:

(i) the risk should be no larger than the maximal net loss

(ii) $R(x, \xi)$ should be subadditive in both variables so that the agent should not want to split his initial capital and the obligations.

(iii) $R(x, \xi_c)$ should be decreasing and convex in x and $x + R(x, \xi_c)$ should be increasing and convex for fixed ξ_c. With this requirement the risk decreases with increasing capital x. Also the risk exposure $\frac{R(x,\xi_c)}{x+R(x,\xi_x)}$ decreases with increasing initial capital.

Obviously, these assumptions are true for the above defined risk measure. First we will now consider the special and simple case of a complete market.

3.1. **The Neyman-Pearson hedge.** We first state some results from Föllmer-Leukert [11]. Consider the class of randomized tests

$$\mathcal{R} := \{\varphi : \Omega \to [0, 1] | \varphi \text{ is } F_T - measurable\}$$

As in this section we live in a complete market any φ may be considered as a contingent claim, when we let φ from L^2 what we assume from now on. Then also $\varphi\xi_c$ is hedgeable and we may state the following result:

Theorem 3.2. *Let $x < p_c(0)$. There exists a $\widetilde{\varphi}$ such that the minimum in*

$$E(p^{\xi_c}(T) - p^{\varphi\xi_c}(T))$$

under the constraint

$$p^{\varphi\xi}(0) \leq x_0$$

is attained in $\widetilde{\varphi}$.

Proof. We follow the technical ideas of Föllmer-Leukert [11]. Let $\mathcal{R}_0$ be the set of randomized tests with $p^{\varphi\xi_c}(0) \leq x_0$. Choose functions $(\widetilde{\varphi}_n) \in \mathcal{R}_0$ in the convex hull of the tail set of a minimizing sequence φ_n such that $\widetilde{\varphi}_n$ converges to some $\widetilde{\varphi} \in \mathcal{R}$.

From Fatou's lemma we then find that the minimum above is attained in $\widetilde{\varphi}$:

$$E\left(p^{\xi_c}(T) - p^{\widetilde{\varphi}\xi_c}(T)\right) \leq E\left(p^{\xi_c}(T) - p^{\varphi\xi_c}(T)\right)$$

for all randomized tests φ and

$p^{\widetilde{\varphi}\xi_c}(0) \leq x_0$. □

Remark 3.3. (i) As φ may be considered as a contingent claim we may look at the bsde

$$dp_t^{\varphi} = rp_t^{\varphi} + \theta q_t^{\varphi}\ dt + qdw_t$$
$$p_T^{\varphi} = \varphi$$

It might be interesting to consider p_t^{φ} as the price of a defautable bond. This will be described in a forthcoming paper.

(ii) Föllmer and Leukert [11], [12] consider a more general case. For a convex function l they consider the problem

$$E\left[l\left(\left(1-\varphi\right)\xi_c\right)\right] = \min_{\varphi}!$$

under

$$\widetilde{E}\left(\varphi\xi_c\right) \leq x_0.$$

To simplify the notation we assume that $\xi_c > 0$. Let $\widetilde{\varphi}$ be the solution in , let $p_t^{\widetilde{\varphi}\xi_c}$ be the price of the knock-out option (see below) $\widetilde{\varphi}\xi_c$ and consider the problem

$$E\left(H_{0T}^r\left(\xi_c - x_T^{x,\pi}\right)^+\right) = \max_{\pi\in A(x)}!$$

for $x \leq x_0 < p_c\left(0\right)$.

Theorem 3.4. *The optimal solution* (x^*, π^*) *of the above problem is given by*

$$x_t^{x^*,\pi^*} \text{ for } \pi^* = \sigma^{-1}q^{\widetilde{\varphi}\xi_c}, x^* = p^{\widetilde{\varphi}\xi_c}\left(0\right).$$

Proof. a) For any admissible strategy (x,π) define the success ratio as

$$\varphi_{(x,\pi)} = 1_{\{x_T^{x,\pi}\geq\xi_c\}} + \frac{x_T^{x,\pi}}{\xi_c}1_{\{x_T^{x,\pi}<\xi_c\}}.$$

Then for any success ratio $\varphi_{(x,\pi)}$

$$p^{\varphi_{(x,\pi)}\xi_c}(0) \leq x^*$$

holds, as the fair price is the infimum of all x such that $(x_t^{x,\pi})$ replicates $x_T^{x,\pi}$:

$$p^{\varphi_{(x,\pi)}\xi_c}(0) \leq p^{x_T^{x,\pi}}(0) \leq x \leq x^{*\cdot}$$

and with $p^{\xi_c} = p_c$

$$E\left(\xi_c - x_T^{x,\pi}\right)^+ = E(p_c(T) - p_c\left(T\right)p^{\varphi_{(x,\pi)}}\left(T\right)) \geq$$
$$E\left(p_c\left(T\right) - p_c\left(T\right)p^{\widetilde{\varphi}}\left(T\right)\right)$$

as $\widetilde{\varphi}$ is optimal for the Neyman-Pearson hedging problem.

b)(x^*, π^*) is obviously admissible. Its success ratio satisfies

$$\varphi_{(x^*,\pi^*)}\xi_c = x_T^{x^*,\pi^*} \wedge \xi_c \geq \widetilde{\varphi}\xi_c \text{ on } \{\xi_c > 0\}$$

and so from 3.1

$$\varphi_{(x^*,\pi^*)}\xi_c = \widetilde{\varphi}\xi_c \; P.a.s. \; on \, \{\xi_c > 0\}.$$

Moreover we have

$$\varphi_{(x^*,\pi^*)} = \widetilde{\varphi} = 1 \text{ on } \xi_c = 0$$

and so

$$\left(\xi_c - x_T^{x^*,\pi^*}\right)^+ = (1-\widetilde{\varphi})\,\xi_c = p_c(T)\left(1 - p^{\widetilde{\varphi}}(T)\right)$$

so that equality holds in 3.1, and so (x^*, π^*) solves the optimization problem □

.

It might be interesting to note that in the complete market case the optimal portfolio automatically is a martingale generating portfolio. The wealth satisfies

$$\begin{gathered} dx_t^{x,\pi} = \left(r(t)\, x_t^{x,\pi} + \theta_j \left(\sigma^*\pi\right)_j\right)dt + \left(\sigma^*\pi\right)_j dw^i \\ x_0^{x,\pi} = x \end{gathered}$$

If the coefficients r, θ, σ are well behaved, then for a $\pi \in A(x)$ the process $(x_t^{x,\pi})$ is a discounted martingale under $\widetilde{P}$. When we denote the -in this sense- martingale generating portfolios by $M(x)$, we here have

$$A(x) = M(x)$$

and a simple consequence is that the optimal portfolio π for the problem

$$E\left[\xi_c - x_T^{x,\pi}\right]^+ = \min_{\pi \in M(x)}!$$

is the same as the optimal portfolio for the problem

$$E\left[\xi_c - x_T^{x,\pi}\right]^2 = \min_{\pi \in A(x)}! = \min_{\pi \in M(x)}! \text{ where } x \leq x_0.$$

From Kohlmann-Zhou [18] we can explicitly compute the optimal portfolio π^* for all initials x:

$$\pi^*(t) = \sigma^{-1}\theta\left(x_t^* - p^*(t)\right) + \sigma^{-1}q^*$$

where $(p^*(t), q^*(t))$ is the Black Scholes price of ξ_c as a solution of the bsde

$$\begin{gathered} dp_t = \left(r(t)\, p_t + \theta_j q_j\right)dt + q_j dw_j \\ p_T = \xi. \end{gathered}$$

and (x_t^*) is the response of the system on applying π^*.

For $x \leq p^*(0)$ we have from the usual comparison theorem that

$$0 \leq p^*(t) - x^*(t) = p^*(0) - x + \int_0^T \left(r\left(p_t^* - x_t^*\right) + \theta^2\left(p_t^* - x_t^*\right) + \theta q^*\right) dt + \int_0^T \left(\theta\left(p_t^* - x_t^*\right) + q^*\right) dw = \left(p^*(t) - x^*(t)\right)^+ = R(x, \xi_c, t)$$

and the minimum is attained in $R(x_0, \xi_c, t)$

Note here that if the equivalent martingale measure coincides with the physical measure the minimal risk is just

$$p^*(0) - x_0$$

and the optimal terminal wealth is

$$(p^*(0) - x_0)P_0(T) + \xi_c$$

which is also obvious from the mean variance approach.

The expectation of $R(x_0, \xi_c, t)$may be interpreted as the expected minimal shortfall at time t, and the success ratio at time t is given by

$$\varphi_{(x^*,\pi^*)}(T) = 1_{\left\{x_T^* \geq p_T^*\right\}} + \frac{x_T^*}{p^*(T)} 1_{\left\{p_T^* > x_T^*\right\}}$$

In the very special case under consideration in this section we can compute the success ratio explicitly in terms of the density (z_{0T}). This is a direct consequence of the Neyman-Pearson lemma:

Theorem 3.5. *The optimal randomized test $\varphi^* \in \mathcal{R}$ is given by*

$$\widetilde{\varphi} = 1_{\left\{z_{0T}^{-1} > a^*\right\}} + \gamma 1_{\left\{z_{0T}^{-1} = a^*\right\}}$$

where

$$a^* = \inf\left\{a^* | \int_{\left\{z_{0T}^{-1} > a\right\}} p_T^* dP = x_0\right\}$$

and

$$\gamma = \left(x^* - \int_{\left\{z_{0T}^{-1} > a^*\right\}} p_T^* dP\right) / \int_{\left\{z_{0T}^{-1} = a^*\right\}} p_T^* dP$$

Proof. The optimization problem

$$E\left(p_T^* - x_t^{x,\pi}\right)^T$$

under the constraint

$$x_0^{x,\pi} \leq x_0$$

is equivalent to

$$E\left(\varphi(T) p^*(T)\right) = \max!$$

under

$$\widetilde{E}\left(\varphi\left(T\right)p^{*}\left(T\right)\right)\leq x_{0}$$

Let us introduce the F_T-densities

$$\frac{dQ}{dP}=\frac{p^{*}(T)}{E(p^{*}(T))}=\frac{E^{*}(\xi_{c}|F_{T})}{E(E^{*}(\xi_{c}|F_{T}))}$$
$$\frac{d\widetilde{Q}}{d\widetilde{P}}=\frac{p*(T)}{\widetilde{E}(p^{*}(T))}=\frac{p^{*}(T)}{\widetilde{E}(\xi_{c})}$$

then the problem is equivalent to

$$\int\varphi\left(T\right)dQ=\max!$$

under

$$\int\varphi\left(T\right)d\widetilde{Q}\leq x=\frac{x_{0}}{\widetilde{E}(\xi_{c})}=\frac{x_{0}}{p^{*}(0)}$$

The Neyman Pearson lemma then implies that the optimal test of the hypothesis Q against the alternative $\widetilde{Q}$ is given by $\widetilde{\varphi}$. □

Remark 3.6. Just some few words for the non-statisticians: Testing two hypotheses H_0 and H_1(the second hypothesis is usually called the anti-hypothesis or alternative) on the basis of an observation is a decision to accept one of the hypotheses as true. In statistics one treats this problem in an unsymmetric way: There are two kinds of mistakes we make when making the decision. First, the observation leads us to accepting H_1 although H_0 is true. This is called an error of first kind. Second a decision for H_0 though H_1 is true is called an error of second kind. Statisticians then find the optimal test as the one which keeps the first kind error below a pre-given boundary and minimizes the error of second kind. The reason for this unsymmetric way of approaching the problem lies in the structure of the problem itself. The probability of a possibly fatal error should be known in advance -in order to keep it small within acceptable bounds- and among all decisions which respect this one looks for the test which minimizes the probability of the second kind error. In our problem the initial investment gives the "error bound" and we minimize the "error" of a shortfall. The details are found in [28].

Remark 3.7. (i) Cvitanic and Karatzas [3] consider an additional requirement on the agent's behaviour, namely that an observer gives a contingent claim which may be thought of as a minimal obligation to be fulfilled certainly

$$\xi_{a}<\xi_{c}$$

with price less than or equal to x_0, the present price of which must not be undergone by $x_{T}^{x,\pi}$. In the present case this problem is trivial as automatically

$$x_T^{x_0,\pi^*} \geq p_a(T)$$

by the well known comparison theorem and the martingale property of π^*.

(ii) It is a simple generalization to consider the problem

$$E(l((\xi_c - x_T^{x,\pi})^+))$$

where l is a concave or convex function. The respective optimal tests are easily computed by making use of the Legendre-Fenchel duals of l.

3.2. **Uncertain real world.** We stay in the market considered in the last sections but now we allow for additional uncertainty in the following sense: The real world measure P is assumed only to belong to a family $\mathcal{P}$.

The problem now consists in finding

$$\underline{V}(x) = \sup_{Q\in\mathcal{P}} \inf_{\pi\in A(x)} E_Q[\xi_c - x_T^{x,\pi}]^+$$

and

$$\overline{V}(x) = \inf_{\pi\in A(x)} \sup_{Q\in\mathcal{P}} E_Q[\xi_c - x_T^{x,\pi}]^+$$

We would now like to define a value of this game by $\overline{V}(x) = \underline{V}(x) = V(x)$.

We shall only shortly describe one way to solve the problem without going into the details as this will be done in a forthcoming paper by making use of the results of the next section.

The idea is to consider

$$k_\nu(\xi_c - x_T^{x,\pi})^+$$

where k_ν is the density of Q with respect to a basic measure P and

$$\mathcal{P} = \{k_v | \nu \in \mathcal{D}\}$$

by abuse of notation. Let us now assume that $k_\nu(\xi_c - x_T^{x,\pi})^+$ is a standard driver for all (ν, π) and let

$$k_{\tilde{\nu}}\left(\xi_c - x_T^{x,\tilde{\nu}}\right)^+ = ess\ \inf_\pi \sup_v \left(k_\nu(\xi_c - x_T^{x,\pi})^+\right).$$

Then a standard result [8] states that Isaacs equation holds:

Theorem 3.8. *Let* $\left(\bar{p}_t, \bar{q}_t\right)$ *be the solution of the bsde with standard driver* $k_{\tilde{\nu}}\left(\xi_c - x_T^{x,\tilde{\nu}}\right)^+$.

Then

$$\begin{aligned}\overline{p}_t &= ess\ \inf_\pi \sup_\nu E(k^\nu (\xi_c - x_T^{x,\pi})^+ |\mathcal{F}_t) \\ &= \sup_\nu ess\ \inf_\pi E(k^\nu (\xi_c - x_T^{x,\pi})^+ |\mathcal{F}_t)\end{aligned}$$

that is we have

$$\overline{V}(x) = \underline{V}(x)$$

and so a price of the contingent claim is well defined in this market.

3.3. **Testing and Optimization.** We reduced the measuring risk problem to solving the Neyman Pearson testing problem

$$E_Q(\varphi) = \max_\varphi!$$

under the constraint

$$E_{\widetilde{Q}}(\varphi) \leq \alpha$$

and in section 3.2 we suggested a way to solve the problem in an uncertain world. Obviously, this problem may be translated into a testing problem of the following kind:

Let P be a family of probability measures and let us assume that all $Q \in P$ are absolutely continuous with respect to a measure μ. For each Q we find (a family of equivalent martingale measures $\widetilde{P} \in \widetilde{P}$ and we may assume that all $\widetilde{P} \in \widetilde{P}$ are also absolutely continuous with respect to μ. So we identify P *with a set* D of densities f_τ, $\widetilde{P}$ with a set D of densities f_τ. Our special assumption here is that we are given a parameter set Θ, disjoint union of Θ_0 and $\Theta_1$such that

$$\begin{aligned}D &= \{f_\tau / \tau \in \Theta_1\} \\ \widetilde{D} &= \left\{\widetilde{f}_\tau / \tau \in \Theta_0\right\}.\end{aligned}$$

This assumption is not as strong as it might seem at first glance. Let us look at a certain world, certain about the physical measure Q, so that Θ_1 contains a single point. Θ_0 then may be seen as a parameter set for the risk premium processes in an incomplete market. If there is only one equivalent martingale measure then also Θ_0 reduces to a set with one single point.

The problem of the last section then takes the form of finding a solution of

$$\max\{\inf_{\tau\in\Theta_1} E_\tau(\varphi) \mid E_\tau(\varphi) \leq \alpha, \tau \in \Theta_0, 0 \leq \varphi \leq 1\}.$$

A solution of this problem is called a max-min Test and standard results from statistics assure the existence for the case when μ is finite. As now we are following a Bayesian approach, we finally assume that Θ_0 and Θ_1 are measurable spaces. The following outline is more or less

taken from a script on "Testtheorie" for a lecture given by Professor Vogel at the University of Bonn in 1974.

Now let ρ and ν be two finite measures on the measurable space (Θ, F) and define the Lagrangian of the above problem by:

$$L(\lambda, \varphi, \nu, \rho, h) =$$
$$\int_{\Theta_0} \alpha(\tau) d\nu + \int_\Omega h d\mu + \lambda(1 - \int_{\Theta_1} d\rho) + \int_\Omega (\int_{\Theta_1} f_\tau d\rho - \int_{\Theta_0} \widetilde{f}_\tau d\nu - h) d\mu$$

for $0 \leq \lambda \in \Re^+, 0 \leq \varphi \leq 1, h \in L_1(\Omega, \mu)$.

From this we find the dual problem for 3.3 as

$$\min \left\{ \int_{\Theta_0} \alpha(\tau) d\nu + \int_\Omega (\int_{\Theta_1} f_\tau d\rho - \int_{\Theta_0} \widetilde{f}_\tau d\nu)^+ d\mu \mid \rho(\Theta_0) = 1, \rho, \nu \geq 0 \right\}.$$

We may now consider the following four equivalent problems:

(S) the Lagrangian has a saddle point $(\lambda_0, \varphi_0, \nu_0, \rho_0, h_0)$

(Kuhn Tucker **KT**) the Kuhn Tucker conditions hold: There is a five-tuple $(\lambda_0, \varphi_0, \nu_0, \rho_0, h_0)$ such that

(i) $\int_{\Theta_1} (E_\tau(\varphi_{0-}) \lambda) d\rho_0 = 0; E_\tau(\varphi_0) \geq \lambda_0, \tau \in \Theta_1$

(ii) $\int_{\Theta_0} (\alpha(\tau) - E_\tau(\varphi_0)) d\nu_0 = 0; E_\tau(\varphi_0) \leq \alpha(\tau), \tau \in \Theta_1$

(iii) $\int_\Omega (1 - \varphi_0) h_0 d\mu = 0, 0 \leq \varphi_0 \leq 1$

(iv) $\int_\Omega (\int_{\Theta_1} f_\tau d\rho_0 - \int_{\Theta_0} f_\tau d\nu_0 - h_0) d\mu = 0, \int_{\Theta_1} f_\tau d\rho_0 - \int_{\Theta_0} f_\tau d\nu_0 \leq h_0$

(v) $\rho_0(\Theta_1) = 1$

(vi) $\varphi_0, \nu_0, \rho_0, h_0 \geq 0$.

(D) a duality formulation holds

$$\max \{\lambda / E_\tau(\varphi) \geq \lambda, \tau \in \Theta_1; E_\tau(\varphi) \leq \alpha(\tau), \tau \in \Theta_0; 0 \leq \varphi \leq 1\}$$
$$=$$
$$\min \left\{ \int_{\Theta_0} \alpha(\tau) d\nu + \int_\Omega h d\mu \mid \int_{\Theta_1} f_\tau d\rho - \int_{\Theta_0} f_\tau d\nu \leq h, \rho(\Theta_1) = 1; \nu, \rho, h \geq 0 \right\}$$

(**NP**) a test has Neyman Pearson structure if

$$\varphi_0 = \begin{bmatrix} 1 & for & \int_{\Theta_1} f_\tau d\rho > \int_{\Theta_0} f_\tau d\nu \\ 0 & for & \int_{\Theta_1} f_\tau d\rho < \int_{\Theta_0} f_\tau d\nu \end{bmatrix}$$

for

$$\int_{\Theta_1} (\alpha(\tau) - E_\tau(\varphi_0)) d\nu_0 = 0, E_\tau(\varphi_0) \leq \alpha(\tau), \tau \in \Theta_0$$
$$\int_{\Theta_0} (E_\tau(\varphi_0) - \lambda_0) d\rho_0 = 0; E_\tau(\varphi_0) \geq \lambda_0, \tau \in \Theta_1$$
$$\rho_0(\Theta_1) = 1, \nu_0, \rho_0 \geq 0.$$

With this we have several equivalent formulations of the problem of measuring risk. The reader familiar with [11], [12] and [3] will recognize the different approaches to the problem as equivalent from the above results and the fact that a least favourable distribution (ν_0, ρ_0) exists if one of the above conditions (S), (KT), (D), (NP) holds. Here (ν_0, ρ_0) is a least favourable distribution if

$$E_{\rho_0}(\varphi_{\rho_0,\nu_0}) \leq E_\rho(\varphi_{\rho,\nu}) \text{ for all } \nu, \rho$$

where E_ρ is the expectation with respect to

$$\int_{\Theta_1} f_\tau d\rho.$$

On the other hand (S), (KT), (D), (NP) hold if a least favourable a priori distribution exists with

$$E_\tau(\varphi_{\rho_0,\nu_0}) \leq \alpha, \tau \in \Theta_1$$

and

$$E_\tau(\varphi_{\rho_0,\nu_0}) \geq \int_{\Theta_1} E_\tau(\varphi_{\rho_0,\nu_0}) d\rho_0, \tau \in \Theta_1.$$

To illustrate the application of these equivalent formulations let us consider the case

$$\#\Theta_i = 1, i = 0, 1.$$

So we have the familiar testing problem

$$\max \{E_{\tau_1}(\varphi)/E_{\tau_0}(\varphi) \leq \alpha, 0 \leq \varphi \leq 1\}$$

which is solved by the standard Neyman-Pearson lemma, namely the optimal test is given by

$$\varphi_0 = \begin{bmatrix} 1 & for & f_{\tau_1} > c f_{\tau_0} \\ 0 & for & f_{\tau_1} < c f_{\tau_0} \end{bmatrix}$$

for some c which has to be determined in the usual way. Equivalently we could solve

$$\min \left\{ \alpha c + \int f_{\tau_1} - c f_{\tau_0})^+ d\mu / c \geq 0 \right\}.$$

The first formulation is the one chosen in [11], the latter was treated in [3].

This duality approach makes it easy to solve the slightly more general problem of measuring risk by

$$\inf_{\pi \in A(x)} E \left[l \left(\frac{\xi - x^{x,\pi}}{P_0(T)} \right)^+ \right]$$

which was shown to be equivalent to

$$\inf_\varphi E \left[l \left(((1 - \varphi)\xi_c)^+ \right) \right]$$

under

$$\widetilde{E}(\varphi \xi_{c)} \leq \alpha,$$

where α was appropriately chosen. We consider two cases:

(i) l is convex and its derivative exists with a well defined inverse I and $l(0) = 0$
(ii) l is concave and bounded, $l(0) = 0$.

In the first case the results on Legendre-Fenchel duals and the transformation of densities show that the optimal Neyman Pearson test is given by

$$\varphi^* = 1 - (I(const\frac{d\widetilde{P}}{dP})/\xi_c \wedge 1)$$

where again we assume that $\xi_c > 0$.
The second case can be reduced to considering the Neyman Pearson lemma of section 3 after replacing ξ_c by $l(\xi_c)$.

This for instance allows to compute the optimal tests φ_p^* for $l(x) = \frac{x^p}{p}$ for both cases $p < 1$ and $p > 1$ where for $p \to 1$ we refind the case treated in the previous sections, for $p \to 0$ the problem becomes the quantile hedging problem

$$P\left(\frac{(\xi_c - x_T^{x,\pi})^+}{P_0(T)}\right) = \min!$$

and for $p \to \infty$ the optimal test is given by

$$\varphi_\infty^* = \frac{(\xi_c - \gamma)^+}{\xi_c}$$

where γ is the constant with

$$c - p^{\gamma\xi_c}(0) = x_0$$

for the case $0 < x_0 < c = p^{\xi_c}(0)$.

When writing this section we became aware of the preprint [13] which exhibits the results in much more detail.

4. Incomplete case

We now go back to section 3.1, here, however, we do no longer assume that the market is complete. So we have to be a bit more careful in stating the analogue of 3.1.

Theorem 4.1. *There exists a test function $\varphi^* \in \mathcal{R}$ which solves the problem*

$$E\left((1-\varphi)\,\xi_c\right) = \min!$$
$$p^{\varphi\xi_c}(0) \leq x_0$$

where

$$dp^{\varphi\xi_c} = rp^{\varphi\xi_c} + \theta\sigma q^{\varphi\xi_c} + \sigma q^{\varphi\xi_c} dw - dC^{\varphi,\xi_c},$$
$$p_T^{\varphi\xi_c} = \varphi\xi_c$$

for (C_s) an increasing continuous process with $C_0 = 0$ is the agent's upper price of the claim.

Proof. The result follows directly from 3.1 and section 2.3.

Now we can follow the outline in the complete market case, and we can state the result in the following form:

Theorem 4.2. *Let $\left(x_0, q^{\varphi\xi_c}, C^{\varphi\xi_c}\right)$ be the strategy determined in 4.1. That is*

$$\left(x_0, q^{\varphi\xi_c}, C^{\varphi\xi_c}\right)$$

is the upper hedging strategy for the contingent claim $\varphi\xi_c$. Then $\left(x_0, \sigma^{-1}q^{\varphi\xi_c}, C^{\varphi\xi_c}\right)$ solves the problem

$$E(\xi_c - x_T^{x,\sigma^{-1}q^{\varphi\xi_c}})^+ = \min_{\pi\in A(x)} (\xi_c - x_T^{x,\pi})^+$$

$x_0 \geq p^{\varphi\xi_c}(0)$.

Proof. The proof follows the same lines as in the complete market case, and so we find the following

Corollary 4.3. *The optimal test here is the optimal test in testing a compound hypothesis against a simple alternative.*

Without further assumptions - of course - we cannot give a more explicit form of the optimal (randomized) test.

5. The American contingent claim

Recall that by definition an American contingent claim has a price which is constrained to be greater than the payoff S_t of the claim at all times $t \in [0, T]$. Even in a perfect market such claims cannot be perfectly hedged by a portfolio. The price process corresponds to the minimal superhedging strategy for the claim. In our notation we have to find a superhedging (self-financing, of course) strategy (π, C). such that

$$x_t^{x,\pi,C} \geq S_t \text{ for all } t$$
$$x_T^{x,\pi,C} = S_T$$

and the infimum of all initial conditions x such that a superhedging strategy exists with the above properties is called the price of the American claim:

$$\inf\{x | \exists_{(\pi,C)} x_t^{\pi,c} \geq S_t, x_T^{\pi,C} = S_T\}.$$

Let us consider the European claims defined in the following way:
We define the set of F_t-stopping times greater or equal to t by

$$\mathcal{I}_{t,T} = \{\sigma / \sigma \text{ is an } F_t\text{-stopping time}, t \leq \sigma \leq T\}$$

For all $\sigma \in \mathcal{I}_{t,T}$ we define the European price of the claim S_σ by $p(t,\sigma)$.
So $p(t,\sigma)$ satisfies

$$dp(t,\sigma) = rp(t,\sigma) + \theta q(t,\sigma)\, dt + q(t,\sigma)dw$$
$$p(\sigma,\sigma) = S_\sigma$$

Conditions which assure that a unique solution of this random duration bsde are given in [2]. The case under consideration here is simpler than the one in [2] as the stopping times under consideration are bounded and so the techniques from the deterministic terminal time case may be adopted.

It is well known [16] that the price of the American claim is now given by the *esssup* of $\{p(t,\sigma)/\sigma \in \mathcal{T}_{tT}\}$. Furthermore [8],[17] the American price is described by a reflected backward stochastic differential equation:

Definition 5.1. Let
(i) $\xi \in L^2$
(ii) a standard generator f
and
(iii) an obstacle $(S_t, 0 \leq t \leq T)$ which is a continuous F_t-adapted process bounded in $L^2, S_T \leq \xi$
be given. By abuse of notation we denote (f, S, ξ)as a standard generator.

We now consider a triple $(p_t, q_t, k_t)_{0 \leq t \leq T}$ of F_t-progressively measurable processes taking values in R, R^n and R_+, respectively, and satisfying
(i) $k \in H^2, p \in H^2, k_T \in L^2$
(ii)$p_t = \xi + \int_t^T f(s, p_s, q_s)\, ds + k_T - k_t - \int_t^T q_s dw_s$
(iii) $p_t \geq S_t, 0 \leq t \leq T$
(iv) (k_t) is continuous and increasing, $k_0 = 0$, and $\int (p_t - S_t)\, dk_t = 0$.
Note that because of (iv) (k_t) pushes the process upwards to stay above S_t. It is only increasing when $k_t = S_t$.

Conditions for well posedness are given in [8], which rely on comparison results similar to the results above. A special difficulty is the construction of the process (k_t). It may be defined by the limit of a sequence of penalized problems[8], [27].

The price of the American claim (S_t, ξ) where without loss of generality we assume $S_T = \xi$, is now given by

$$\begin{gathered} dp_t = (rp_t + \theta q_t)\, dt + q_t dw_t - dk_t \\ p_t \geq S_t, p_T = S_T. \end{gathered}$$

Now let us consider the situation that an agent is faced the obligation to be able to pay S_t at any time $t \in [0, T]$. He will be able to do so if his initial capital is greater than or equal to p_0. If he is unwilling or unable to invest this amount at time $t = 0$, say he only has the capital $x_0 < p_0$, then we are in a similar situation as in section 3.

So we consider the European claim S_v for a stopping time $\nu \in \mathcal{I}_{0T}$. Then from chapter 3 we know that there is a test function such that

$$\min_{\pi} E\left(S_\nu - x_\nu^{x,\pi}\right)^{+} = E\left(\left(1 - \varphi\left(\nu\right)\right) S_v\right)$$

where

$$x = p_0^{\varphi,v}, \pi = \sigma^{-1} q^{\varphi,v}$$

is given by

$$\begin{gathered} dp_t^{\varphi,v} = \left(rp_t^{\varphi,\nu} + \theta q_t^{\varphi,\nu}\right) dt + q_t^{\varphi,\nu} dw_t \\ p_v^{\varphi,\nu} = \varphi\left(\nu\right) S_v. \end{gathered}$$

From the above consideration it is clear that the following proposition holds

Theorem 5.2. *Let $x_0 < p_0$. The optimal strategy $(x_0, \pi^\varphi, k^\varphi)$ is given as the solution of the rbsde*

$$\begin{gathered} dp_t^\varphi = \left(rp^\varphi + \theta p^\varphi\right) dt + q^\varphi dw - dk^\varphi \\ p_t^\varphi \geq \varphi\left(t\right) S_t,\ p_T^\varphi = \varphi\left(T\right) S_T,\ \int \left(p_t^\varphi - \varphi\left(t\right) S_t\right) dk^\varphi = 0 \end{gathered}$$

6. Conclusion

We might consider a completely different approach to solve the problems considered in this paper. Let us assume that again we are in a complete market. As we have seen above the claim hedgeable from the starting point x may be assumed to be x_T^{x,π^*} under the assumptions of section 3. Let again $x < p^{\xi_c}(0)$ for a contingent claim ξ_c. Then the solution of

$$\begin{gathered} dp_t^{\xi_c} = (rp_t^{\xi_c} + \theta q^{\xi_c}) dt + q^{\xi_c} dw \\ p_T^{\xi_c} = \xi_c \end{gathered}$$

may be considered as a supersolution of

$$dp_t^{x_T^{x,\pi^*}} = dp_t^* = (rp_t^* + \theta q_t^*)dt + q^* dw_t$$
$$p_T^* = x_T^{x\pi^*}$$

Then there is a right continuous process (C_t), such that $\left(p_t^{\xi_c}, q_t^{\xi_c}, C_t\right)$solves

$$dp_t^* = (rp_t^* + \theta q_t^*)dt + q^* dw_t - dC_t$$
$$p_T^* = x_T^{x,\pi^*}.$$

Obviously, C_T is just the amount of shortfall, namely

$$C_T = p_T^{\xi_c} - p_T^{\xi_c} \wedge p_T^*$$

As the infimum of the two solutions is a solution

$$\widetilde{p}_t^{x,\pi}$$

of the problem

$$\left(f, \xi_c \wedge x_T^{x,\pi^*}\right)$$

so that

$$C_T = p_T^{\xi_c} - \widetilde{p}_T^{x,\pi}.$$

In this framework the quantile hedging problem may be formulated in the following form:

Consider the simply coupled FBSDE

$$dx_t^{x,\pi} = \left(r_t x_t^{x,\pi} + \theta_t \sigma\pi\right) dt + \sigma\pi dw, x_0^{x,\pi} = x$$
$$dp_t = (rp_t + \theta q_t)dt + q dw_t - dC_t$$
$$p_T = x_T^{x\pi}$$

Solve this problem for any π, compute the supersolution $(\widetilde{p}_t^{x,\pi}, \widetilde{q}_t^{x,\pi}, \widetilde{C}^{x,\pi})$ and find a pair (x^*, π^*) such that

$$E(\widetilde{p}_t^{x,\pi}) = \max_\pi$$

with respect to

$$\widetilde{p}^{x,\pi}(0) \leq x.$$

Then the shortfall is simply given by

$$E^*(\xi_c) - x + \widetilde{C}^{x^*,\pi^*}(T).$$

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