

LINEAR QUADRATIC OPTIMIZATION FOR MODELS WITH RATIONAL EXPECTATIONS

HANS M. AMMAN AND DAVID A. KENDRICK

Abstract. In this paper we present a method for using rational expectations in a linear-quadratic optimization framework. Following the approach put forward by Sims, we solve the model through a QZ decomposition, which is generally easier to implement than the more widely used method of Blanchard and Kahn.

1. Introduction

Starting with the work of Kydland and Prescott (1976) and Lucas (1976), much criticism was made about the use of control theory in economics. One of the major drawbacks of the use of the so called classical control theory is that it cannot deal with rational expectations. There are a number of generic methods to solve models with rational expectations (RE). For instance, Fair and Taylor (1983) use an iterative method for solving RE models and, in the tradition of Theil, Fisher, Holly and Hughes Hallett (1986) uses a method based on stacking the model variables. A hybrid method based on the saddlepoint property is presented in Anderson and Moore (1985).

The Blanchard and Kahn method (1980) is another well known method for solving linear models with RE in discrete time. By decomposing the model into stable and unstable parts, the unstable part can be solved forward in time and the stable part backward in time. Although the BK approach is theoretically a powerful method, for practical implementation it has some serious drawbacks. First, not all linear RE models can be put into first order linear form as required by the Blanchard and Kahn method, and second, the method is based on the *Jordan canonical form*. The procedure to put the model into the Jordan canonical is not widely available in software libraries and is known to be numerically unstable, see Moler (1994).

In recent work, Amman, Kendrick and Achath (1995), Amman (1996), we presented a procedure that introduces RE in a linear-quadratic (LQ) control framework based on the Blanchard and Kahn method. Due to the limitations of the Blanchard and Kahn approach and the fact that we had to rely on the diagonalization of the transition matrix, this work could only deal with a limited set of models. Recently, Sims (1996) proposed a different method for solving linear models with RE allowing for a broader range of models. This method is not based on the Jordan canonical form, but uses the more widely available *QZ* form that is based on generalized eigenvalues and which is more

Date: September 23, 1997.

Key words and phrases. Macroeconomics, Rational Expectations, stochastic optimization, numerical experiments.

JEL Classification: C63, E61. Corresponding author: Hans M. Amman, Department of Economics and Tinbergen Institute, University of Amsterdam, Roetersstraat 11, Room E1-913, 1018 WB Amsterdam, the Netherlands, Email amman@fee.uva.nl.

numerically stable. In this paper we will follow the paper of Sims and incorporate his approach into the LQ framework.

2. Problem statement and solution

Following Kendrick (1981), the standard single-agent linear-quadratic optimization problem is written as:

Find the set of admissible instruments $U = \{u_0, u_1, \dots, u_{T-1}\}$ that minimizes the welfare loss function

$$(1) \quad J_T = \beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t)$$

with

$$\begin{aligned} L_T &= \frac{1}{2}(x_T - \bar{x}_T)' W_T (x_T - \bar{x}_T) \\ L_t &= \frac{1}{2}(x_t - \bar{x}_t)' W_t (x_t - \bar{x}_t) + \\ &\quad \frac{1}{2}(u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t) + (x_t - \bar{x}_t)' F_t (u_t - \bar{u}_t) \end{aligned}$$

subject to the model

$$(2) \quad x_{t+1} = A_t x_t + B_t u_t + C_t z_t$$

The vector $x_t \in \mathfrak{R}^n$ is the state of the economy at time t and the vector $u_t \in \mathfrak{R}^m$ contains the policy instruments. The initial state of the economy x_0 is known, \bar{x}_t and \bar{u}_t are target values. W_t , R_t and F_t are penalty matrices of conformable size, β being a discount factor.

The above model is straightforward to solve and there are a number of packages available, see Amman and Kendrick (1997a). However, a serious drawback for economics is that equation (2) does not allow for rational expectations. One way of allowing RE to enter the model is to augment equation (2) in the following fashion

$$(2a) \quad x_{t+1} = A_t x_t + B_t u_t + C_t z_t + \sum_{j=1}^k D_{j,t} E_t x_{t+j} + \epsilon_t$$

where the matrix $D_{j,t}$ is a parameter matrix, $E_t x_{t+1}$ is the expected state for time $t + j$ at time t , k the maximum lead in the expectations formation and ϵ_t is a white noise vector. In order to compute the admissible set of instruments we have to eliminate the rational expectations from the model. In a previous paper Amman, Kendrick and Achath (1995), we used the Blanchard and Kahn method to solve the RE in the model. However, as mentioned above, the Blanchard and Kahn method has some features that impede practical implementation. First, the method requires that the model may be put into first order linear form, so D_k should be invertible. Second, the BK approach uses the *Jordan canonical form method*. This method is applicable to any transition matrix. However, it is not widely available in software libraries and is known to be numerically

unstable.

Sims (1996) proposes a different route by applying a *generalized eigenvalue* approach. In order to apply Sims' method we first put equation (2) in the form¹

$$(3) \quad \Gamma_{0,t}\tilde{x}_{t+1} = \Gamma_{1,t}\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t$$

where

$$\Gamma_{0,t} = \begin{bmatrix} I - D_{1,t} & -D_{2,t} & \dots & -D_{k-1,t} & -D_{k,t} \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & & I & 0 \end{bmatrix}$$

$$\Gamma_{1,t} = \begin{bmatrix} A_t & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & & I \\ 0 & \dots & & 0 \end{bmatrix} \quad \Gamma_{2,t} = \begin{bmatrix} B_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_{3,t} = \begin{bmatrix} C_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Gamma_4 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and the augmented state vector

$$(4) \quad \tilde{x}_t = \begin{bmatrix} x_t \\ Ex_{t+1} \\ Ex_{t+2} \\ \vdots \\ Ex_{t+k-1} \end{bmatrix}$$

Taking the generalized eigenvalues of equation (3) allows us to decompose the system matrices $\Gamma_{0,t}$ and $\Gamma_{1,t}$ in the following manner, *viz.* Coleman and Van Loan (1988) or Moler and Stewart (1973) ,

$$\Lambda_t = Q_t\Gamma_{0,t}Z_t$$

$$\Omega_t = Q_t\Gamma_{1,t}Z_t$$

with $Z_t'Z_t = I$ and $Q_t'Q_t = I$. The matrices Λ_t and Ω_t are upper triangular matrices and the generalized eigen values are $\forall i \omega_{i,i}/\lambda_{i,i}$. If we use the transformation $w_t = Z_t'\tilde{x}_t$ and $w_{t+1} = Z_t'\tilde{x}_{t+1}$ we can write equation (3) as

$$(5) \quad \Lambda_t w_{t+1} = \Omega_t w_t + Q_t\Gamma_{2,t}u_t + Q_t\Gamma_{3,t}z_t + Q_t\Gamma_4\epsilon_t$$

¹Note that in contrast to Sims (1996) the variable z_t contains exogenous variables and not random variables. Hence, the matrix Π in Sims' paper is set to zero.

It is possible to reorder the matrices Z_t and Q_t in such a fashion that the diagonal elements of the matrices Λ_t and Ω_t contain the generalized eigenvalues in ascending order. In that case we can write equation (4) as follows

$$(6) \quad \begin{bmatrix} \Lambda_{11,t} & \Lambda_{12,t} \\ 0 & \Lambda_{22,t} \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Omega_{11,t} & \Omega_{12,t} \\ 0 & \Omega_{22,t} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{2,t} u_t + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{3,t} z_t + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_4 \epsilon_t$$

where the unstable eigen values are in lower right corner, that is the matrices $\Lambda_{22,t}$ and $\Omega_{22,t}$. By forward propagation and taking expectations, it is possible to derive $w_{2,t}$ as a function of future instruments and exogenous variables, Sims (1996, page 5)

$$(7) \quad \gamma_t = w_{2,t} = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \Omega_{22,t+j}^{-1} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j} + \Gamma_{3,t+j} z_{t+j})$$

The matrix $\tilde{M}_{t,j}$ is defined as

$$\tilde{M}_{t,j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for } j > 0$$

and

$$\tilde{M}_{t,j} = I \quad \text{for } j = 0$$

with

$$M_t = \Omega_{22,t}^{-1} \Lambda_{22,t}$$

Given the fact that $\Omega_{22,t}$ contains the eigenvalues outside the unit circle, we have applied the follow condition in deriving equation (7)

$$\lim_{j \rightarrow \infty} \tilde{M}_{t,j} = 0$$

In contrast to Sims, M_t is not time invariant since we explicitly want to allow for time varying parameters in the model. Reinserting equation (7) into equation (6) gives us

$$(8) \quad \tilde{\Lambda}_t w_{t+1} = \tilde{\Omega}_t w_t + \tilde{\Gamma}_{2,t} u_t + \tilde{\Gamma}_{3,t} z_t + \tilde{\Gamma}_{4,t} \epsilon_t + \tilde{\gamma}_t$$

with

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{11,t} & \Lambda_{12,t} \\ 0 & I \end{bmatrix} \quad \tilde{\Omega} = \begin{bmatrix} \Omega_{11,t} & \Omega_{12,t} \\ 0 & 0 \end{bmatrix} \quad \tilde{\Gamma}_{2,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{2,t}$$

$$\tilde{\Gamma}_{3,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{3,t} \quad \tilde{\Gamma}_{4,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_4 \quad \tilde{\gamma}_t = \begin{bmatrix} 0 \\ \gamma_t \end{bmatrix}$$

Knowing that $\tilde{x}_t = Z_t w_t$ and $\tilde{x}_{t+1} = Z_t w_{t+1}$ we can write equation (8) as

$$(9) \quad \tilde{x}_{t+1} = \tilde{A}_t \tilde{x}_t + \tilde{B}_t u_t + \tilde{C}_t \tilde{z}_t + \tilde{\epsilon}_t$$

with

$$(10) \quad \tilde{A}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Omega}_t Z_t' \quad \tilde{B}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{2,t} \quad \tilde{C}_t = [Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{3,t} \quad Z_t \tilde{\Lambda}_t^{-1}]$$

and

$$(11) \quad \tilde{\Lambda}_t^{-1} = \begin{bmatrix} \Lambda_{11,t}^{-1} & -\Lambda_{11,t}^{-1} \Lambda_{12,t} \\ 0 & I \end{bmatrix} \quad \tilde{z}_t = \begin{bmatrix} z_t \\ \tilde{\gamma}_t \end{bmatrix} \quad \tilde{\epsilon}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{4,t} \epsilon_t$$

We have to make the assumption here that $\Lambda_{11,t}$ is nonsingular. However, the diagonal elements will generally be nonzero, so it is very likely that the matrix is nonsingular. With equation (9) we have transformed equation (2a) into the form required by equation (2) enabling us to set up an iterative scheme using the LQ framework in equations (1)-(2). Knowing that γ_t depends on the future the basic algorithm works like this ²

Step 0. Set the iteration count $\nu = 0$ and set the instruments u_t^ν , $t = \{0, 1, \dots, T+s-1\}$, set \tilde{x}_0 (see below).

Step 1. Compute γ_t^ν , $t = \{0, 1, \dots, T+s-1\}$, by using

$$\gamma_t^\nu = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \Omega_{22,t+j}^{-1} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j}^\nu + \Gamma_{3,t+j} z_{t+j})$$

Step 2. Apply the LQ framework in equations (1)-(2) to compute a new set of optimal instruments $u_t^{\nu+1}$ using the equation below in place of equation (2)

$$\tilde{x}_{t+1}^{\nu+1} = \tilde{A}_t \tilde{x}_t^{\nu+1} + \tilde{B}_t u_t^{\nu+1} + \tilde{C}_t \tilde{z}_t^\nu$$

Step 3. Set $\nu = \nu + 1$ and go to Step 1 until convergence has been reached.

There is still one remaining issue we have to take care of. In order to apply the algorithm for $\nu = 0$ we need to have an initial value of the augmented state vector, which is

$$(12) \quad \tilde{x}_0 = \begin{bmatrix} x_0 \\ Ex_1 \\ Ex_2 \\ \vdots \\ Ex_{k-1} \end{bmatrix}$$

However, at the beginning of the first iteration step the expectational variables are not known, so we have to give the expectational part of \tilde{x}_0^0 an arbitrary value. Once we have completed the first iterations we can use the computed states to reinitialize \tilde{x}_0^1 . In the next section we will apply this algorithm using a simple macro model.

²The Matlab implementation of this algorithm can be obtained through the corresponding author.

3. An example

Consider a simple macro model with output, x_t , consumption, c_t , investment, i_t , government expenditures, g_t , and taxes τ_t . The problem can then be stated as:

Find for the model

$$(13) \quad x_{t+1} = c_{t+1} + i_{t+1} + g_{t+1}$$

$$(14) \quad c_{t+1} = 0.8(x_t - \tau_t) + 200$$

$$(15) \quad i_{t+1} = 0.2E_t x_{t+2} + 100 + \epsilon_t$$

$$(16) \quad g_{t+1} = u_t$$

$$(17) \quad \tau_{t+1} = 0.25x_{t+1}$$

with $x_0 = 1500$, a set of admissible control $U = \{u_0, u_1, \dots, u_9\}$ to minimize the welfare loss function

$$(18) \quad J_T = \frac{1}{2}(x_{10} - 1600)^2 + \frac{1}{2} \sum_{t=0}^9 \{(x_t - 1600)^2 + g_t^2\}$$

If we reduce the above model to one equation for output we get

$$(19) \quad x_{t+1} = 0.6x_t + u_t + 0.2E_t x_{t+2} + 300 + \epsilon_t$$

which leads to the augmented system

$$(20) \quad \begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}$$

where $z_t = 1 \forall t$. Applying the QZ factorization, Coleman and van Loan (1988), to compute the generalized eigen values of the model gives us the time invariant solution

$$(21) \quad \Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix} \quad \Omega = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7951 \end{bmatrix}$$

$$(22) \quad Z = \begin{bmatrix} 0.8202 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix} \quad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix}$$

so the eigenvalues are $\{0.7546/1.0822, 0.7952/0.1848\} = \{0.6972, 4.328\}$ and apparently the ordering of the system is such that the unstable root 4.328 is in the lower right corner of Λ and Ω . The other components are

$$(23) \quad \tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0.4944 \\ 0.3447 \end{bmatrix}$$

$$(24) \quad \tilde{C} = \begin{bmatrix} 148.3243 & 0.7580 & 0.1205 \\ 103.4153 & 0.5285 & 1.3031 \end{bmatrix}$$

$$(25) \quad \tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = [1] \quad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(26) \quad \tilde{x}_0 = \begin{bmatrix} 1500 \\ 1500 \end{bmatrix} \quad \tilde{\tilde{x}}_0 = \begin{bmatrix} 1600 \\ 0 \end{bmatrix} \quad \bar{u} = [0]$$

Note that we have set $E_0 x_1^0 = 1500$. For this simple model we can compute the steady state of the system being $x_\infty = 1589.08$ and $u_\infty = 17.81$, see Amman and Kendrick (1997b). Hence, $U^0 = \{17.81, \dots, 17.81\}$ is an educated guess for the instruments. The solution to the model is in Table 1.

Table 1.
Solution of the LQ optimization model with RE

t	0	1	2	3	4	5	6	7	8	9	10
x_t	1500	1556	1576	1584	1587	1588	1589	1589	1587	1584	1578
u_t	40	26	21	19	18	18	18	17	16	11	

4. Summary

In this paper we have presented a single agent Linear-Quadratic optimization model that allows for rational expectations. Based on Sims's paper we have used a generalized eigenvalue method for solving the variables that involve the unstable roots. By using an iterative scheme, the reduced model can be fitted into a standard Linear-Quadratic framework that allows us to derive the optimal policy instruments for the model with rational expectations.

References

- [1] Amman, H.M., 1990, Implementing adaptive control software on supercomputing machines, *Journal of Economics, Dynamics and Control* **14**, 265-279.
- [2] Amman, H.M., 1996, Numerical Optimization Methods for Dynamic Optimization Problems. In H.M. Amman, D.A. Kendrick and J. Rust (editors), *Handbook of Computational Economics*, 579-618. Appeared in the series *Handbooks in Economics* edited by K.J. Arrow and M.D. Intriligator, North-Holland Publishers.
- [3] Amman, H.M. D.A. Kendrick, 1997a, The DUALI/DUALPC Software for optimal control models. In H.M. Amman, B. Rustem and A.B. Whinston (editors), *Computational Approaches to Economic Problems*, Advances in Computational Economics, Kluwer Academic Publishers, 363-372.
- [4] Amman, H.M. D.A. Kendrick, 1997b, A note on computing the steady state of the linear-quadratic optimization model with rational expectations, *Research Memorandum*, Department of Economics, University of Amsterdam.
- [5] Amman, H.M. and H. Neudecker, 1997, Numerical solutions of the algebraic matrix Ricatti equation, *Journal of Economic Dynamics and Control* **21**, 363-369.
- [6] Amman, H.M., D.A. Kendrick and S. Achath, 1995, Solving stochastic optimization models with learning and rational expectations, *Economics Letters* **48**, 9-13.
- [7] Anderson, G. and G. Moore, 1985, A linear algebraic procedure for solving linear perfect foresight models, *Economics Letters* **17**, 247-252.
- [8] Blanchard, O.J. and C.M. Kahn, 1980, The solution of linear difference models under rational expectations, *Econometrica* **48**, 1305-1311.
- [9] Coleman, T.F. and C. van Loan, 1988, *Handbook for matrix computations*, SIAM, Philadelphia.
- [10] Fair, R.C. and J. Taylor, 1993, Solution and maximum likelihood estimation of dynamic rational expectations models, *Econometrica* **52**, 1169-1185.
- [11] Fisher, P.G., S. Holly and A.J. Hughes Hallett, 1986, Efficient solution techniques for dynamic non-linear rational expectations models, *Journal of Economic Dynamics and Control* **10**, 139-145.

- [12] Kendrick, D.A., 1981, *Stochastic Control for Economic Models*, McGraw-Hill, New York.
- [13] Kydland, F.E. and E.C. Prescott, 1997, Rules are than discretion: the time inconsistency of optimal plans, *Journal of Political Economy* **85**, 473-491.
- [14] Lucas, R.E., 1996, Econometric policy evaluation: A critique. In K. Brunner and A.H. Meltzer (editors), *The Phillips curve and the labor markets*, 19-46. Supplementary series to the *Journal of Monetary Economics*.
- [15] Moler, C.B., 1994, Jordan's canonical form just doesn't compute, *The MathWorks Newsletter* **8**, 8-9.
- [16] Moler, C.B. and G.W. Stewart, 1973, An algorithm for for generalized matrix eigenvalue problems, *SIAM Journal of Numerical Analysis* **10**, 241-256.
- [17] Sims, C. A, 1996, Solving linear rational expectations models, *Research paper*, Department of Economics, Yale University.

Appendix A Derivation of Equation 5

Begin with equation (3), i.e.

$$(A-1) \quad \Gamma_{0,t}\tilde{x}_{t+1} = \Gamma_{1,t}\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t$$

From the equation just above equation (5)

$$(A-2) \quad \Lambda_t = Q_t\Gamma_{0,t}Z_t$$

$$(A-3) \quad \Omega_t = Q_t\Gamma_{1,t}Z_t$$

Premultiply equations (A-2) and (A-3) by Q'_t and post multiply them by Z'_t keeping in mind that $Z'_tZ_t = I$ and $Q'_tQ_t = I$ yields

$$(A-4) \quad Q'_t\Lambda_tZ'_t = \Gamma_{0,t}$$

$$(A-5) \quad Q'_t\Omega_tZ'_t = \Gamma_{1,t}$$

Then substitute equations (A-4) and (A-5) into equation (A-1) to obtain

$$(A-6) \quad Q'_t\Lambda_tZ'_t\tilde{x}_{t+1} = Q'_t\Omega_tZ'_t\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t$$

Premultiplication of equation (A-6) by Q_t and use of the definitions

$$(A-7) \quad w_{t+1} = Z'_t\tilde{x}_{t+1}$$

$$(A-8) \quad w_t = Z'_t\tilde{x}_t$$

yields

$$(A-9) \quad \Lambda_t w_{t+1} = \Omega_t w_t + Q_t\Gamma_{2,t}u_t + Q_t\Gamma_{3,t}z_t + Q_t\Gamma_4\epsilon_t$$

which is equation (5) in the body of the paper.

Appendix B
Derivation of Equation 7

Begin with the bottom half of equation (6), i.e.

$$(B-1) \quad \Lambda_{22,t} w_{2,t+1} = \Omega_{22,t} w_{2,t} + Q_{2t} \Gamma_{2,t} u_t + Q_{2t} \Gamma_{3,t} z_t + Q_{2t} \Gamma_{4,t} \epsilon_t$$

For the sake of simplification define

$$(B-2) \quad \xi_t = Q_{2t} (\Gamma_{2,t} u_t + \Gamma_{3,t} z_t + \Gamma_{4,t} \epsilon_t)$$

Then using equation (B-2) in equation (B-1) and rearranging terms we obtain

$$(B-3) \quad \Omega_{22,t} w_{2,t} = \Lambda_{22,t} w_{2,t+1} - \xi_t$$

or

$$(B-4) \quad w_{2,t} = \Omega_{22,t}^{-1} \Lambda_{22,t} w_{2,t+1} - \Omega_{22,t}^{-1} \xi_t$$

Then define

$$(B-5) \quad M_t = \Omega_{22,t}^{-1} \Lambda_{22,t}$$

and use equation (B-5) in equation (B-4) to obtain

$$(B-6) \quad w_{2,t} = M_t w_{2,t+1} - \Omega_{22,t}^{-1} \xi_t$$

Next solve equation (B-6). Begin this process by using equation (B-6) to obtain

$$(B-7) \quad w_{2,t+1} = M_{t+1} w_{2,t+2} - \Omega_{22,t+1}^{-1} \xi_{t+1}$$

Then substitute equation (B-7) into equation (B-6) to obtain

$$(B-8) \quad w_{2,t} = M_t (M_{t+1} w_{2,t+2} - \Omega_{22,t+1}^{-1} \xi_{t+1}) - \Omega_{22,t}^{-1} \xi_t$$

or

$$(B-9) \quad w_{2,t} = M_t M_{t+1} w_{2,t+2} - M_t \Omega_{22,t+1}^{-1} \xi_{t+1} - \Omega_{22,t}^{-1} \xi_t$$

Repeat the process above for $w_{2,t+2}$. First, from equation (B-7)

$$(B-10) \quad w_{2,t+2} = M_{t+2} w_{2,t+3} - \Omega_{22,t+2}^{-1} \xi_{t+2}$$

Then substitution of equation (B-10) into equation (B-9) yields

$$(B-11) \quad w_{2,t} = M_t M_{t+1} (M_{t+2} w_{2,t+3} - \Omega_{22,t+2}^{-1} \xi_{t+2}) - M_t \Omega_{22,t+1}^{-1} \xi_{t+1} - \Omega_{22,t}^{-1} \xi_t$$

or

$$(B-12) \quad w_{2,t} = M_t M_{t+1} M_{t+2} w_{2,t+3} - M_t M_{t+1} \Omega_{22,t+2}^{-1} \xi_{t+2} - M_t \Omega_{22,t+1}^{-1} \xi_{t+1} - \Omega_{22,t}^{-1} \xi_t$$

This process can be continued for s periods until one obtains

$$(B-13) \quad w_{2,t} = \left(\prod_{i=0}^s M_{t+i} \right) w_{2,t+s+1} - \sum_{j=1}^s \left(\prod_{i=0}^{j-1} M_{t+i} \right) \Omega_{22,t+j}^{-1} \xi_{t+j} - \Omega_{22,t}^{-1} \xi_t$$

Under the condition that

$$(B-14) \quad \lim_{s \rightarrow \infty} \left(\prod_{i=0}^s M_{t+i} \right) = 0$$

equation (B-13) becomes in the limit as $s \rightarrow \infty$

$$(B-15) \quad w_{2,t} = - \sum_{j=1}^{\infty} \left(\prod_{i=0}^{j-1} M_{t+i} \right) \Omega_{22,t+j}^{-1} \xi_{t+j} - \Omega_{22,t}^{-1} \xi_t$$

Then define

$$(B-16) \quad \tilde{M}_{t,j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for } j \geq 1$$

and

$$(B-17) \quad \tilde{M}_{t,j} = I \quad j = 0$$

and write equation (B-15) as

$$(B-18) \quad w_{2,t} = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \Omega_{22,t+j}^{-1} \xi_{t+j}$$

Substitution of equation (B-2) into equation (B-18) then yields after taking expectations

$$(B-19) \quad w_{2,t} = - \sum_{j=0}^{\infty} \tilde{M}_{t,j} \Omega_{22,t+j}^{-1} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j} + \Gamma_{3,t+j} z_{t+j})$$

which is the same as equation (7) in the body of the paper.

Department of Economics and Tinbergen Institute, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, the Netherlands.

E-mail address: amman@fee.uva.nl

Department of Economics, University of Texas, Austin, Texas 78712, USA.

E-mail address: kendrick@eco.utexas.edu