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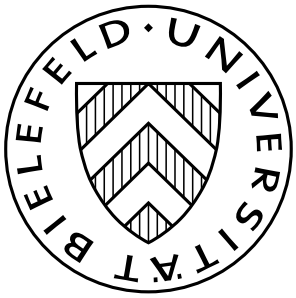

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Weak approximation of G-expectation*

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Abstract

We refine the discretization of G-expectation by Y. Dolinsky, M. Nutz, and M. Soner (*Stochastic Processes and their Applications, 122 (2012), 664–675*), in order to obtain a discretization of sublinear expectation where the martingale laws are defined on a finite lattice rather than the whole set of reals.

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Keywords: G-expectation; Volatility uncertainty; Weak limit theorem.

1 Introduction

In [5], *Dolinsky et al* showed a Donsker-type result for G-Brownian motion, henceforth referred to as *G-Donsker,* by introducing a notion of volatility uncertainty in discrete time and defined a discrete version of *Peng's* G-expectation. In the continuous-time limit, the resulting sublinear expectation converges weakly to G-expectation. In their discretization, *Dolinsky et al.* [5] allow for martingale laws whose support is the whole set of reals. In other words, they only discretize the time line, but not the state space of the canonical process. Now for certain applications, for example a hyperfinite construction of G-expectation in the sense of Robinsonian nonstandard analysis, a discretization of the state space would be necessary. We will show in this paper, that a slight modification of the construction by *Dolinsky et al.* [5] suffices to obtain a discretization where the state space for the discrete-time canonical process is discretized, too (whence the martingale

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laws are supported by a finite lattice only). We will prove the convergence of this discretization to continuous-time G-expectation.

This paper is organised as follows: in Section 2, we introduce G-expectation, and the discrete-time and continuous-time version of the sublinear expectation in the spirit of [5]. Unlike in [5], we require the discretization of the martingale laws to be defined on a finite lattice rather than the whole set of reals. In Section 3, we introduce the strong formulation of volatility uncertainty, and show that a natural push forward of our discretize sublinear expectation converges weakly to G-expectation as $n \to \infty$. Finally, we prove that

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] = \lim_{n \to \infty} \max_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})].$$

2 Framework

2.1 G-expectation via volatility uncertainty

Peng [9, 10] introduced a sublinear expectation on a well-defined space $\mathbb{L}^1_G$, the completion of $\mathrm{Lip}_{b.cyl}(\Omega)$ (bounded and Lipschitz cylinder function) under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, under which the increments of the canonical process $(B_t)_{t>0}$ are zero-mean, independent and stationary and can be proved to be (G)-normally distributed (see [8]). This type of process is called *G-Brownian motion* and the corresponding sublinear expectation is called *G-expectation*. We fix a constant $T > 0$ and the dimension $d = 1$.

The G-expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear operator defined on a class of random variables on Ω. The symbol G refers to a given function $G : \mathbb{R} \to \mathbb{R}$ of the form

$$G(\gamma) = \frac{1}{2}(R\gamma^+ - r\gamma^-) = \frac{1}{2} \sup_{c \in [r,R]} c\gamma,$$

where $0 \leq r \leq R < \infty$ are fixed numbers. The construction of the G-expectation is as follows. Let $\xi = f(B_T)$, where B_T is the G-Brownian motion and f a sufficiently regular function. Then $\mathcal{E}^G(\xi)$ is defined to be the initial value $u(0,0)$ of the solution of nonlinear backward heat equation,

$$\partial_t u - G(\partial^2_{xx} u) = 0,$$

with terminal condition $u(\cdot, T) = f$, [7]. The mapping $\mathcal{E}^G$ can be extended to random variables of the form $\xi = f(B_{t_1}, \cdots, B_{t_n})$ by a stepwise evaluation of the PDE and then to the completion $\mathbb{L}^1_G$ of the space of all such random variables. [3] showed that $\mathbb{L}^1_G$ is the completion of $C_b(\Omega)$ and $\mathrm{Lip}_{b.cyl}(\Omega)$ under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, and that $\mathbb{L}^1_G$ is the space of the so-called quasi-continuous function and contains all bounded continuous function on the canonical space Ω (however, not all bounded measurable functions are included). Proposition 3.1 cannot be extended to the case where ξ is merely in $\mathbb{L}^1_G$ under the norm $\|\cdot\|_{\mathbb{L}^1_G}$, because $\|\cdot\|_{\mathbb{L}^1_G}$ does not see the discrete-time

object (see [5], Remark 3.2), thus, we work in a smaller space $\mathbb{L}^1_*$ defined as the completion of $C_b(\Omega;\mathbb{R})$ under the norm $\|\cdot\|_*$.

There also exists an alternative representation of G-expectation known as the dual view on G-expectation via volatility uncertainty [3–5]: One can show that the G-expectation can be expressed as the upper expectation

$$\mathcal{E}^G(\xi) = \sup_{P\in\mathcal{P}^G} \mathbb{E}^P[\xi], \quad \xi = f(B_T), \tag{1}$$

wherein $\mathcal{P}^G$ is defined as the set of probability measures on Ω such that, for any $P \in \mathcal{P}^G$, B is a martingale with the volatility $d\langle B\rangle_t/dt \in \mathbf{D}$ in $P \otimes dt$ a.e, and $\mathbf{D} = [r, R]$, for $0 \le r \le R < \infty$.

2.2 Continuous-time construction of sublinear expectation

Let $\Omega = \{\omega \in \mathcal{C}([0,T];\mathbb{R}) : \omega_0 = 0\}$ be the canonical space of continuous paths with time horizon $T \in (0,\infty)$, endowed with uniform norm $\|\omega\|_\infty = \sup_{0\le t\le T}|\omega_t|$, where the Euclidean norm on $\mathbb{R}$ is given by $|\cdot|$. Let B be the canonical process $B_t(\omega) = \omega_t$, and $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$ is the filtration generated by B. A probability measure P on Ω is called a martingale law of B, if B is P-martingale. Then, $\mathcal{P}_{\mathbf{D}}$ is the set of martingale laws on Ω and the volatility takes values in $\mathbf{D}$, $P \otimes dt$ a.e;

$$\mathcal{P}_{\mathbf{D}} = \{P\text{-martingale law on } \Omega;\ d\langle B\rangle_t/dt \in \mathbf{D},\ P\otimes dt \text{ a.e.}\}.$$

Thus, the sublinear expectation is given by

$$\mathcal{E}_{\mathbf{D}}(\xi) = \sup_{P\in\mathcal{P}_{\mathbf{D}}} \mathbb{E}^P[\xi], \text{ such that, for any } \xi : \Omega \to \mathbb{R}, \tag{2}$$

ξ is $\mathcal{F}_T$-measurable and integrable for all $P \in \mathcal{P}_{\mathbf{D}}$.

2.3 Discrete-time construction of sublinear expectation

Here, we introduce the sublinear expectation, $\mathcal{E}^n$, on the n-step canonical space where $n \in \mathbb{N}$. We denote,

$$\mathcal{L}_n = \left\{ \frac{j}{\sqrt{n}}, \quad -n \le j \le n, \quad \text{for } j \in \mathbb{Z} \right\},$$

where $\mathcal{L}_n^{n+1} = \mathcal{L}_n \times \cdots \times \mathcal{L}_n (n+1 \text{ times})$. Let $X^n = (X_k^n)_{k=0}^n$ be the canonical process $X_k^n(x) = x_k$ defined on $\mathcal{L}_n^{n+1}$ and $(\mathcal{F}_k^n)_{k=0}^n = \sigma(X_l^n, l = 0,\dots,k)$ be the filtration generated by X^n, such that, $\mathbf{D}$ is a bounded set of volatilities. A probability measure P on $\mathcal{L}_n^{n+1}$ is a martingale law if X^n is a P-martingale and $X_0^n = 0$ P a.s. The increment ΔX^n is denoted by $\Delta X_k^n = X_k^n - X_{k-1}^n$. Then, $\mathcal{P}_{\mathbf{D}}^n$ is a set of martingale laws on $\mathcal{L}_n^{n+1}$, such that, the volatility takes values in $\mathbf{D}$, $P \otimes dt$ a.e;

$$\mathcal{P}_{\mathbf{D}}^n = \left\{P\text{-martingale law on } \mathcal{L}_n^{n+1};\ \mathbb{E}^P[(\Delta X_k^n)^2] \in \mathbf{D} \text{ and } r_{\mathbf{D}} \le |\Delta X_k^n|^2 \le R_{\mathbf{D}},\ P \text{ a.e.}\right\}.$$

Let $\mathbb{P}^n_{\mathbf{D}}$ be the set of martingale laws on $\mathbb{R}^{n+1}$, i.e.,

$$\mathbb{P}^n_{\mathbf{D}} = \left\{P\text{-martingale law on } \mathbb{R}^{n+1};\ \mathbb{E}^P[(\Delta X^n_k)^2] \in \mathbf{D} \text{ and } r_{\mathbf{D}} \leq |\Delta X^n_k|^2 \leq R_{\mathbf{D}},\ P \text{ a.e.}\right\},$$

such that for all n, $\mathcal{L}^{n+1}_n \subseteq \mathbb{R}^{n+1}$ and $\mathcal{P}^n_{\mathbf{D}} \subseteq \mathbb{P}^n_{\mathbf{D}}$. Thus, the sublinear expectation is given by

$$\mathcal{E}^n_{\mathbf{D}}(\psi) = \sup_{P \in \mathcal{P}^n_{\mathbf{D}}} \mathbb{E}^P[\psi], \text{ such that, for any } \psi : \mathcal{L}^{n+1}_n \to \mathbb{R}, \tag{3}$$

ψ is $\mathcal{F}^n_n$- adapted for all $P \in \mathcal{P}^n_{\mathbf{D}}$.

In order to establish a relation between the two formulations, we obtained a continuous-time process $x = \{\widehat{x}_t : 0 \leq t \leq T\}$ from any discrete path $x \in \mathcal{L}^{n+1}_n$ by linear interpolation. i.e.,

$$\widehat{x} := ([nt/T] + 1 - nt/T)x_{[nt/T]} + (nt/T - [nt/T])x_{[nt/T]+1}$$

where $\widehat{\ }: \mathcal{L}^{n+1}_n \to \Omega$ is the linear interpolated operator, $x = (x_0, \ldots, x_n) \mapsto \widehat{x} = \{(\widehat{x})_{0 \leq t \leq T}\}$, and $[y]$ denote the greatest integer less than or equal to y. If X^n is the canonical process on $\mathcal{L}^{n+1}_n$ and ξ is a random variable on Ω, then $\xi(\widehat{X^n})$ defines a random variable on $\mathcal{L}^{n+1}_n$. Thus,

$$\mathcal{E}^n_{\mathbf{D}}(\xi(\widehat{X^n})) = \sup_{P \in \mathcal{P}^n_{\mathbf{D}}} \mathbb{E}^P[\xi(\widehat{X^n})]$$

can be seen as the discrete analogue of G-expectation, and this allows us to extend $\mathcal{E}^n_{\mathbf{D}}(\psi)$ to a continuous-time object.

3 Results and proofs

We introduce the so-called strong formulation of volatility uncertainty for the discrete-time construction, as in [5], and for the continuous-time construction, as in [5, 6, 11, 12]; i.e., we consider martingale laws generated by stochastic integrals with respect to a fixed random walk and Brownian motion. For the continuous-time construction; let $\mathcal{Q}_{\mathbf{D}}$ be the set of martingale laws of the form;

$$\mathcal{Q}_{\mathbf{D}} = \left\{ P_0 \circ (M)^{-1};\ M = \int f(t, B) dB_t, \text{ and } f \in \mathcal{C}([0,T] \times \Omega; \mathbf{D}^{1/2}) \text{ is adapted} \right\}.$$

B is the canonical process under the Wiener measure P_0, and $\mathbf{D}$ is a convex set. Recall from [5], (Proposition 3.5,) that the convex hull of $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{P}_{\mathbf{D}}$. For the discrete-time construction; we fix $n \in \mathbb{N}$, $\Omega_n = \{\omega = (\omega_1, \ldots, \omega_n) : \omega_i \in \{\pm 1\}, \quad i = 1, \ldots, n\}$ equipped with the power set and let $P_n = \underbrace{\{2^{-1}, \ldots, 2^{-1}\}}_{\text{n times}}$ be the product probability associated with the uniform

distribution. Let $\xi_1, \ldots, \xi_n$ be an i.i.d sequence of $\mathcal{L}_n$-valued random variable such that ξ_k belongs to $\{\pm 1\}$ and the components of ξ_k are orthonormal in $L^2(P_n)$. We denote the associated random walk by $Z_k^n = \sum_{l=1}^k \xi_l$, then, we can view

$$\sum_{l=0}^{k-1} f(l, \mathbb{X}) \Delta \mathbb{X}_l$$

as discrete-time integrals of $\mathbb{X}$, where f is $\mathcal{F}^n$-adapted and $\mathbb{X} = \frac{1}{\sqrt{n}} Z^n$ is the scaled random walk. We denote by $\mathcal{Q}^n_{\mathbf{D}/n}$ the set of martingale laws of the form;

$$\mathcal{Q}^n_{\mathbf{D}/n} = \left\{ \begin{array}{l} P_n \circ (M^f)^{-1};\ f : \{0, \ldots, n\} \times \mathcal{L}_n^{n+1} \to \mathbf{D}_n^{1/2} \text{ is } \mathcal{F}^n\text{-adapted}, \\ \text{and } \forall\ k \in \{0, \ldots, n\},\ M_k^f = \sum_{l=0}^{k-1} f(l, \mathbb{X}) \Delta \mathbb{X}_l, \end{array} \right\}.$$

Proposition 3.1 states that this sublinear expectation with discrete-time volatility uncertainty on a finite space converges to G-expectation.

Proposition 3.1. *Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \| \omega \|_\infty)^b$ for some constants $a, b > 0$. Then,*

$$\lim_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] = \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{4}$$

Proof. To prove (4), we prove two separate inequalities together with a density argument which imply (4). Before then, we introduce a smaller space $\mathbb{L}^1_*$ that is defined as the completion of $\mathcal{C}_b(\Omega; \mathbb{R})$ under the norm

$$\| \xi \|_* := \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q |\xi|, \quad \mathcal{Q} := \left\{ P \in \mathcal{P}_{\mathbf{D}} \right\} \cup \left\{ P \circ \widehat{(X^n)}^{-1}; P \in \mathbb{P}^n_{\mathbf{D}/n}, n \in \mathbb{N}. \right\}.$$

[5], (Lemma 3.4,) show that if $\xi : \Omega \to \mathbb{R}$ satisfies the condition of Proposition 3.1, then $\xi \in \mathbb{L}^1_*$.

First inequality (for $\leq$ in (4))**:**

$$\limsup_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \leq \sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi].$$

Let $\mathcal{Q}^n_{\mathbf{D}}$ be the set of martingale laws of the form;

$$\mathcal{Q}^n_{\mathbf{D}} = \left\{ P_n \circ \left(\sum_{l=0}^{k-1} f(l, Z) \Delta Z_l \right)^{-1};\ f : \{0, \ldots, n\} \times \mathbb{R}^{n+1} \to \mathbf{D}^{1/2} \text{ is adapted} \right\}.$$

[5] show that $\mathcal{Q}^n_{\mathbf{D}} \subseteq \mathbb{P}^n_{\mathbf{D}}$. For all n, trivially $\mathcal{L}_n^{n+1} \subseteq \mathbb{R}^{n+1}$. Thus, $\mathcal{Q}^n_{\mathbf{D}/n} \subseteq \mathcal{Q}^n_{\mathbf{D}}$. Therefore,

$$\sup_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \leq \sup_{P \in \mathcal{Q}^n_{\mathbf{D}}} \mathbb{E}^P[\xi(\widehat{X^n})],$$

and for all n, we have

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \leq \limsup_{n\to\infty} \sup_{P\in\mathcal{Q}^n_{\mathbf{D}}} \mathbb{E}^{P}[\xi(\widehat{X^n})]. \tag{5}$$

In [5], it was shown that

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{Q}^n_{\mathbf{D}}} \mathbb{E}^{P}[\xi(\widehat{X^n})] \leq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi].$$

Thus,

$$\limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \leq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi]. \tag{6}$$

Second inequality (for $\geq$ in (4))**:**

It remains to show that

$$\liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \geq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^{P}[\xi].$$

For arbitrary $P \in \mathcal{Q}_{\mathbf{D}}$, we construct a sequence $(P^n)_n$ such that for all n,

$$P^n \in \mathcal{Q}^n_{\mathbf{D}/n}, \tag{7}$$

and

$$\mathbb{E}^{P}[\xi] \leq \liminf_{n\to\infty} \mathbb{E}^{P^n}[\xi(\widehat{X^n})]. \tag{8}$$

Fix n, and let $\xi_1, \ldots, \xi_n$ be some i.i.d sequence of random variables on Ω_n as defined above, i.e., $\xi_i : \Omega_n \to \mathcal{L}_n$, for $i = 1, \ldots, n$. Now, we want to construct martingales M^n whose laws are in $\mathcal{Q}^n_{\mathbf{D}/n}$ and the laws of their interpolations tend to P. To achieve the above task, we introduce a scaled random walk with the piecewise constant càdlàg property (right continuity with left limits),

$$W^n_t := \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt/T]} \xi_k, \quad 0 \leq t \leq T, \tag{9}$$

and we denote the continuous version of (9) obtained by linear interpolation by

$$\widehat{W^n_t} := \frac{1}{\sqrt{n}} \widehat{Z^n}, \quad 0 \leq t \leq T. \tag{10}$$

By the central limit theorem;

$(W^n, \widehat{W^n})$ converges in distribution to a Brownian motion (W, W),

i.e., the law (P_n) converges to the law P_0 on the Skorohod space $D([0,T];\mathbb{R}^2)$ [2]. Let $g \in \mathcal{C}([0,T]\times\Omega, \mathbf{D}^{1/2})$, such that

$$P = P_0 \circ \left(\int g(t,W) dW_t = M \right)^{-1}.$$

Since $\widehat{W_t^n}$ is the interpolated version of (9), it turns out that

$$g\left([nt/T], \widehat{W^n}\right) \text{ converges in distribution to } g(t,W) \text{ on } D([0,T];\mathbb{R}).$$

We introduce martingales with discrete-time integrals,

$$M_k^n := \sum_{l=0}^{k-1} g\left(\frac{lT}{n}, \widehat{W^n}\right) \frac{1}{\sqrt{n}} \left(Z_{l+1}^n - Z_l^n\right). \tag{11}$$

In order to show that $P_n \circ (M^n)^{-1} \in \mathcal{Q}^n_{\mathbf{D}/n}$, we have to find

$$g_n : \{0,\dots,n\} \times \mathcal{L}_n^{n+1} \to \mathbf{D}^{1/2}/n, \quad \forall\, l \in \{0,\dots,n\},$$

such that,

$$M_k^n = \sum_{l=0}^{k-1} g_n\left(l, \frac{1}{\sqrt{n}} Z^n\right) \frac{1}{\sqrt{n}} \Delta Z_l^n.$$

This is possible: simply put

$$g_n(\ell, \vec{\mathbb{X}}) \mapsto g\left(\frac{\ell T}{n}, \widehat{\vec{W}}\right).$$

Therefore,

$$P_n \circ (M^n)^{-1} \in \mathcal{Q}^n_{\mathbf{D}/n}.$$

By [5], the continuous version of (11) obtained by linear interpolation $\widehat{M^n}$ converges in distribution to M on Ω endowed with the uniform metric on the Skorohod space, i.e., $\widehat{M^n} \Rightarrow M$ on Ω. Since ξ is bounded and continuous,

$$\lim_{n\to\infty} \mathbb{E}^{P_n\circ(M^n)^{-1}}[\xi(\widehat{X^n})] = \mathbb{E}^{P_0\circ M^{-1}}[\xi]. \tag{12}$$

Therefore, (7) is satisfied for $P^n = P_n \circ (M^n)^{-1} \in \mathcal{Q}^n_{\mathbf{D}/n}$. Trivially, (7) implies

$$\mathbb{E}^{P^n}[\xi(\widehat{X^n})] \le \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})]. \tag{13}$$

Combining (12) and (13), and taking the lim inf as n tends to ∞, gives

$$\mathbb{E}^{P}[\xi] \le \liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})]. \tag{14}$$

Taking the supremum of (14) over $P \in \mathcal{Q}_{\mathbf{D}}$, the equation becomes

$$\sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] \leq \liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})].$$

Hence,

$$\liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \geq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{15}$$

Combining (6) and (15),

$$\begin{aligned}\sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] &\geq \limsup_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \\ &\geq \liminf_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \\ &\geq \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi].\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] = \sup_{P\in\mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi]. \tag{16}$$

Density argument: Hence (4) is established for all $\xi \in \mathcal{C}_b(\Omega, \mathbb{R})$. Since $\mathcal{Q}^n_{\mathbf{D}/n} \subseteq \mathcal{Q}$ and $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{Q}$, this implies that $\mathcal{Q}_{\mathbf{D}} \subseteq \mathcal{P}_{\mathbf{D}}$ [5], (Propostion 3.5,) that is (4) holds for all $\xi \in \mathbb{L}^1_*$, and hence [5], (Lemma 3.4,) holds for all ξ that satisfy condition of Proposition 3.1. □

Proposition 3.2. *Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some constants $a, b > 0$ and $Q^n_{D/n}$ be the set of probability measure as defined in* (18)*, then*

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] = \max_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})]. \tag{17}$$

Proof. The LHS of (17) can be written as

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] = \sup_{f\in\mathcal{A}} \mathbb{E}^{P_n\circ(M^f)^{-1}}[\xi(\widehat{X^n})],$$

where $\mathcal{A} = \{f : \{0, \ldots, n\} \times \mathcal{L}^{n+1}_n \to \mathbf{D}^{1/2}\}$ such that f is $\mathcal{F}^n$-adapted. We shall prove that $\mathcal{A}$ is a compact subset of a finite-dimensional vector space, and that $f \mapsto \mathbb{E}^{P_n\circ(M^f)^{-1}}[\xi(\widehat{X^n})]$ is continuous.

First part

Recall; for fix $n \in \mathbb{N}$, $X^n = (X^n_k)^n_{k=0}$ is the canonical process defined by $X^n_k(x) = x_k$ for $x = (x_0, \ldots, x_n) \in \mathcal{L}^{n+1}_n$, and $(\mathcal{F}^n_k)^n_{k=0} = \sigma(X^n_l, l = 0, \ldots, k)$ is the filtration generated by X^n. We consider $\Omega_n = \{\omega =$

$(\omega_1, \ldots, \omega_n) : \omega_i \in \{\pm 1\},\ i = 1, \ldots, n\}$ equipped with the power set, and $P_n = \{2^{-1}, \ldots, 2^{-1}\}^n$ is the product probability associated with the uniform distribution. $\xi_1, \ldots, \xi_n$ is the i.i.d sequence of real-valued random variable such that ξ_k belongs to $\{\pm 1\}$ and the components of ξ_k are orthonormal in $L^2(P_n)$. We denote the associated random walk by $Z_k^n = \sum_{l=1}^k \xi_l$.
$\mathcal{A}$ is closed[1] and obviously bounded with respect to the norm $\|\cdot\|_\infty$ as $\mathbf{D}$ is bounded[2]. By Heine-Borel theorem, $\mathcal{A}$ is a compact subset of a $N(n,n)$-dimensional vector space equipped with the norm $\|\cdot\|_\infty$.

Second part

Here, we want to show that $F : f \mapsto \mathbb{E}^{P_n \circ (M^f)^{-1}}[\xi(\widehat{X^n})]$ is continuous.

$$\mathcal{Q}^n_{\mathbf{D}/n} = \left\{ \begin{array}{l} P_n \circ (M^f)^{-1};\ f : \{0, \ldots, n\} \times \mathcal{L}_n^{n+1} \to \mathbf{D}^{1/2} \text{ is } \mathcal{F}^n\text{-adapted,} \\ \text{and } \forall\ k \in \{0, \ldots, n\},\ M_l^f = \sum_{l=0}^{k-1} f(l, \mathbb{X}) \Delta \mathbb{X}_l, \end{array} \right\}, \tag{18}$$

and

$$\mathbb{X} = \frac{1}{\sqrt{n}} Z^n.$$

$$\begin{aligned} \mathbb{E}^{P_n \circ (M^f)^{-1}}[\xi(\widehat{X^n})] &= \int_{\mathcal{L}_n^{n+1}} \xi(\widehat{X^n}) dP_n \circ (M^f)^{-1}, \\ &= \int_{\Omega_n} \xi(\widehat{X^n}(M^f)) dP_n, \quad \text{(transforming measure)} \\ &= \sum_{\omega_n \in \Omega_n} P_n\{\omega_n\} \xi \circ (\widehat{X^n}) \circ M^f(\omega_n). \end{aligned}$$

From Proposition 3.1 we know that ξ is continuous, $\widehat{X^n}$ is the interpolated canonical process, i.e., $\widehat{X} : \mathcal{L}_n^{n+1} \to \Omega$, thus $\widehat{X^n}$ is continuous and P_n takes it value from the set of real numbers. For $F : f \mapsto \mathbb{E}^{P_n \circ (M^f)^{-1}}[\xi(\widehat{X^n})]$ to be continuous, $\psi : f \mapsto M^f$ has to be continuous. Since $\mathcal{A} = \{f : \{0, \ldots, n\} \times \mathcal{L}_n^{n+1} \to \mathbf{D}^{1/2}$, where f is adapted with respect to the filtration generated by $\mathbb{X}\}$ is a compact subset of a $N(n,n)$-dimensional vector space for fix $n \in \mathbb{N}$ and $M^f : \Omega_n \to \mathcal{L}_n^{n+1}$, for all $f, g \in \mathcal{A}$,

$$|M^f - M^g| = |\|f\|_\infty - \|g\|_\infty| \leq \|f - g\|_\infty.$$

[1]The cardinality of $\mathcal{L}_n$, $\#\mathcal{L}_n = 2n+1$, $\#\mathcal{L}_n^{n+1} = (2n+1)^{n+1}$, and $\#(\{0, \ldots, n\} \times \mathcal{L}_n^{n+1}) = (n+1)(2n+1)^{n+1} = N(n,n)$. Let $(f^m)_m \in \mathcal{A}^{N(n,n)}$ and $f : \{0, \ldots, n\} \times \mathcal{L}_n^{n+1} \to \mathbb{R}$, such that $f^m \to f$, as $m \to \infty$, with respect to the maximium norm $\|\cdot\|_\infty$ (or any norm as a result of norm equivalent) on $\mathbb{R}^{N(n,n)}$. We have to prove that f is adapted and $D^{1/2}$-valued (is obvious, $D^{1/2}$ is closed). For the first part, let $j \in \{0, \ldots, n\}$. We want to show that $f(j, \cdot)$ is $\mathcal{F}_j^n$-measurable. This, however, follows from ([1], Theorem 13.4(ii).).

[2]If $V \in \mathbb{R}_{>0}$ such that $D \subseteq [0, V]$, then obviously $\|f\|_\infty = \max_{\substack{j \in \{0, \ldots, n\} \\ \omega \in \mathcal{L}_n^{n+1}}} |f(j, \omega)| \leq \sqrt{V}$.

Thus, ψ is continuous with respect to the norm $\|\cdot\|_\infty$. Hence F is continuous with respect to any norm[3] on $\mathbb{R}^{N(n,n)}$.

□

Theorem 3.1. *Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some constants $a, b > 0$. Then,*

$$\sup_{P \in \mathcal{Q}_{\mathbf{D}}} \mathbb{E}^P[\xi] = \lim_{n \to \infty} \max_{\mathbb{Q} \in \mathcal{Q}^n_{\mathbf{D}/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})]. \tag{19}$$

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[3] For any two vector norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, and $C_1, C_2 > 0$, we have $C_1\|A\|_\alpha \leq \|A\|_\beta \leq C_2\|A\|_\alpha$, for all matrices $A \in \mathbb{R}^{N(n,n)}$. i.e., all norms on $\mathbb{R}^{N(n,n)}$ are equivalent because $\mathbb{R}^{N(n,n)}$ has $N(n,n)$-dimension for fix $n \in \mathbb{N}$.

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Appendix

Density argument verification

Let

$$f : \xi \mapsto \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi]$$

and

$$g : \xi \mapsto \lim_{n \to \infty} \sup_{\mathbb{Q} \in \mathcal{Q}_D^n/n} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})].$$

From (16), we know that for all $\xi \in \mathcal{C}_b(\Omega, \mathbb{R})$, $f(\xi) = g(\xi)$. Since $\mathbb{L}_*^1$ is the completion of $\mathcal{C}_b(\Omega, \mathbb{R})$ under the norm $\|\cdot\|_*$, $\mathcal{C}_b(\Omega, \mathbb{R})$ is dense in $\mathbb{L}_*^1$; and we want to prove for all $\xi \in \mathbb{L}_*^1$, $f(\xi) = g(\xi)$. To prove this, it is sufficient to show that f and g are continuous with respect to the norm $\|\cdot\|_*$.

For continuity of f:

$\forall P \in \mathcal{Q}_D$ and $\xi, \xi^{'} \in \mathbb{L}_*^1$,

$$\sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi^{'}] \leq \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi - \xi^{'}]$$

and

$$\sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi - \xi^{'}] \leq \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[|\xi - \xi^{'}|].$$

Since, $\mathcal{Q}_D \subseteq \mathcal{Q}$,

$$\sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[|\xi - \xi^{'}|] \leq \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[|\xi - \xi^{'}|] = \|\xi - \xi^{'}\|_*.$$

Then,

$$\sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi^{'}] \leq \|\xi - \xi^{'}\|_*. \tag{20}$$

Interchanging ξ and $\xi^{'}$,

$$\sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi^{'}] - \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi] \leq \|\xi^{'} - \xi\|_*. \tag{21}$$

Adding (20) and (21), we have

$$\left| \sup_{P\in\mathcal{Q}_D} \mathbb{E}^P[\xi] - \sup_{P\in\mathcal{Q}_D} \mathbb{E}^P[\xi^{'}] \right| \leq \|\xi - \xi^{'}\|_*. \tag{22}$$

Hence,

$$|f(\xi) - f(\xi^{'})| \leq \|\xi - \xi^{'}\|_*.$$

For continuity of g:

Since $\mathcal{Q}^n_{D/n} \subseteq \mathcal{Q}$, we can follow the same argument as above; for all $\mathbb{Q} \in \mathcal{Q}^n_{D/n}$, $\xi, \xi^{'} \in \mathbb{L}^1_*$ and for all n,

$$\begin{aligned}
&\sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] - \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X^n})] \\
&\qquad \leq \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n}) - \xi^{'}(\widehat{X^n})] \\
&\qquad \leq \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[|\xi(\widehat{X^n}) - \xi^{'}(\widehat{X^n})|].
\end{aligned}$$

Since, $\mathcal{Q}^n_{D/n} \subseteq \mathcal{Q}$, we can say that

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[|\xi(\widehat{X^n}) - \xi^{'}(\widehat{X^n})|] \leq \sup_{Q\in\mathcal{Q}} \mathbb{E}^{Q}[|\xi - \xi^{'}|] = \|\xi - \xi^{'}\|_*,$$

then,

$$\sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] - \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X^n})] \leq \|\xi - \xi^{'}\|_*. \tag{23}$$

When the limit of n tends to ∞, (23) becomes,

$$\lim_{n\mapsto\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] - \lim_{n\mapsto\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X^n})] \leq \|\xi - \xi^{'}\|_*. \tag{24}$$

Interchanging ξ and $\xi^{'}$,

$$\lim_{n\mapsto\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X^n})] - \lim_{n\mapsto\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] \leq \|\xi^{'} - \xi\|_*. \tag{25}$$

Adding (24) and (25), we have

$$\left| \lim_{n\mapsto\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi(\widehat{X^n})] - \lim_{n\mapsto\infty} \sup_{\mathbb{Q}\in\mathcal{Q}^n_{D/n}} \mathbb{E}^{\mathbb{Q}}[\xi^{'}(\widehat{X^n})] \right| \leq \|\xi - \xi^{'}\|_*.$$

Hence,

$$\left| g(\xi) - g(\xi^{'}) \right| \leq \|\xi - \xi^{'}\|_*.$$