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Autoregressive Aided Periodogram Bootstrap for Time Series

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Abstract

A bootstrap methodology for the periodogram of a stationary process is proposed which is based on a combination of a time domain parametric and a frequency domain nonparametric bootstrap. The parametric fit is used to generate periodogram ordinates that imitate the essential features of the data and the weak dependence structure of the periodogram while a nonparametric (kernel based) correction is applied in order to catch features not represented by the parametric fit. The asymptotic theory developed shows validity of the proposed bootstrap procedure for a large class of periodogram statistics. For important classes of stochastic processes, validity of the new procedure is established also for periodogram statistics not captured by existing frequency domain bootstrap methods based on independent periodogram replicates.

AMS 1990 subject classifications: Primary 62G09; secondary 62M10

Key words: Bootstrap, periodogram, nonparametric estimators, ratio statistics, spectral means.

1. Introduction

Consider a strictly stationary univariate process $\mathbf{X} = (X_t : t \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\})$ and assume that X_t has the representation

$$X_t \;=\; \sigma \sum_{\nu=0}^{\infty} \alpha_\nu \varepsilon_{t-\nu} \;,\; t \in \mathbb{Z} \tag{1.1}$$

where $\{\alpha_\nu\}$, $\alpha_0 = 1$, is an absolute summable sequence, $\{\varepsilon_t\}$ is a sequence of independent, identically distributed random variables with mean zero and unit variance and σ a positive constant. Assume that we have observations $X_1, X_2, ..., X_n$ of the process $\mathbf{X}$ at hand. Statistical inference in the frequency domain is commonly based on the so-called periodogram $I_n(\lambda)$,

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{-i\lambda t} \right|^2 \;,\; \lambda \in [0, \pi] \tag{1.2}$$

which is known to be an asymptotically unbiased but not consistent estimator of the spectral density f of the process $\mathbf{X}$.

Because of (1.1) and assumption (A1) below f has the representation

$$f(\lambda) = \frac{\sigma^2}{2\pi} \left| 1 + \sum_{\nu=1}^{\infty} \alpha_\nu e^{-i\nu\lambda} \right|^2 \;,\; \lambda \in [0, \pi] \;. \tag{1.3}$$

In the sequel we require in some cases invertibility of $\{X_t\}$ which narrows the class (1.1) a little bit. Invertibility ensures that the process $\{X_t\}$ can be represented as a one-sided infinite order autoregressive process

$$X_t \;=\; \sum_{\nu=1}^{\infty} a_\nu X_{t-\nu} + \sigma \cdot \varepsilon_t \;,\; t \in \mathbb{Z} \tag{1.4}$$

where $\{a_\nu\}$ is an absolute summable sequence.

Methods of bootstrapping the periodogram $I_n(\lambda)$ have attracted considerable attention in recent years. Compared to time domain bootstrap methods, the appeal of frequency domain methods lie on the fact that for a huge class of stochastic processes, the observed series $X_1, X_2, \ldots, X_n$ can be transformed into a set of $N = [n/2]$ nearly independent statistics, the periodogram ordinates at the so-called Fourier frequencies $\lambda_j = 2\pi j/n$, $j = 0, 1, 2, \ldots, N$. Since for $\lambda_j \in (0, \pi)$ the mean and the variance of $I_n(\lambda_j)$ are approximately equal to $f(\lambda_j)$ and $f^2(\lambda_j)$ respectively, bootstrap methods designed for a nonparametric regression setup with independent errors can be potentially applied to bootstrap the periodogram.

For Gaussian processes, frequency domain bootstrap methods have been considered among others by Nordgaard (1992) and Theiler et al. (1994). Using the property that the relation between periodogram and spectral density can be approximately described by means of a multiplicative regression model, Franke and Härdle (1992) proposed a nonparametric

bootstrap method based on an initial (nonparametric) estimate of the spectral density f and i.i.d. resampling of (appropriately defined) frequency domain residuals. Franke and Härdle established asymptotic validity of their method for nonparametric (kernel) estimators of the spectral density while Dahlhaus and Janas (1996) extended the validity of this bootstrap procedure to the class of the so-called ratio statistics and to Whittle estimators. An alternative idea to bootstrap the periodogram has been proposed by Paparoditis and Politis (1999). Their method uses smoothness properties of the spectral density f and the periodogram replicates are obtained by locally resampling from adjacent periodogram ordinates.

The independence of the bootstrap periodogram ordinates is an essential feature of the bootstrap procedures mentioned above that restricts the classes of statistics to which the existing methods can be successfully applied. Loosely speaking, validity of the above nonparametric bootstrap procedures can be established only for periodogram statistics for which the weak and asymptotically vanishing dependence of the periodogram ordinates does not affect their large-sample distribution. Nonparametric estimators and ratio statistics have this property; cf. Franke and Härdle (1992) and Dahlhaus and Janas (1996). However, there are other interesting classes of periodogram statistics for which the dependencies of the periodogram ordinates sum up to a non vanishing contribution. For instance, Dahlhaus (1985) investigated the following class of integrated periodogram estimators

$$\int_o^\pi \varphi(\lambda) I_n(\lambda) d\lambda \tag{1.5}$$

for some appropriate defined functions φ on $[0, \pi]$. The following are some special cases covered by (1.5).

Example:

(i) $\varphi(\lambda) = 2\cos(\lambda h)$, $h \in \mathbb{N}_o$, leads to the empirical autocovariance $\hat{\gamma}_n(h)$, since

$$\int_0^\pi 2\cos(\lambda h) I_n(\lambda) d\lambda = \frac{1}{n} \sum_{t=1}^{n-h} X_t X_{t+h} =: \hat{\gamma}_n(h) \ ,$$

which is a consistent and $\sqrt{n}-$consistent estimate of the true underlying autocovariance $\gamma(h) = EX_t X_{t+h}$.

(ii) $\varphi(\lambda) = 1_{[0,x]}(\lambda)$, $x \in [0, \pi]$ leads to the integrated periodogram

$$F_n(x) = \int_0^x I_n(\lambda) d\lambda \ ,$$

which consistently estimates the spectral distribution function $F(x) = \int_0^x f(\lambda) d\lambda$.

Again from Dahlhaus (1985) it is known, that under suitable assumptions the asymptotic distribution of

$$\sqrt{n} \left(\int_o^\pi \varphi(\lambda) I_n(\lambda) d\lambda - \int_o^\pi \varphi(\lambda) f(\lambda) d\lambda \right) \tag{1.6}$$

is Gaussian with mean zero and variance given by

$$2\pi \int_o^\pi \varphi^2(\lambda) f^2(\lambda) d\lambda + \kappa_4 \left(\int_o^\pi \varphi(\lambda) f(\lambda) d\lambda \right)^2 \tag{1.7}$$

where κ_4 is the fourth cumulant of ε_t. Note that instead of (1.5) a discretized version may also be used. Discretization of the integral is usually done along the Fourier-frequencies $\lambda_j = 2\pi j/n, j = 0, 1, ..., N$. This leads to the following discrete version of (1.6)

$$\frac{2\pi}{\sqrt{n}} \sum_{j=0}^{N} \varphi(\lambda_j) \{ I_n(\lambda_j) - f(\lambda_j) \} d\lambda \ . \tag{1.8}$$

Under some smoothness assumptions on φ the difference of (1.6) and (1.8) is asymptotically negligible.

A modification of the Franke/Härdle frequency domain bootstrap has been proposed by Janas and Dahlhaus (1994) in order to deal with the periodogram estimators (1.5). However, their method is based on a direct estimation of the fourth order cumulant of the error process which requires nonparametric estimation of functionals of the spectral density of $\{X_t\}$ and of the squared process $\{X_t^2\}$.

In this paper we introduce a new bootstrap procedure for the periodogram which is based on a combination of a parametric time domain and a nonparametric frequency domain bootstrap. The essential feature of the new bootstrap proposal is the following: For periodogram statistics for which the dependence of the periodogram ordinates does not affect their asymptotic distribution, the bootstrap procedure proposed 'works' under the same set of process assumptions as those required for the aforementioned fully nonparametric methods. Furthermore, for stochastic processes possessing representation (1.4), our procedure leads also to asymptotically valid approximations for more general classes of periodogram statistics including those given by (1.5).

It is mentioned above, that in case we are interested in ratio statistics, i.e.

$$\frac{\int_0^\pi \varphi(\lambda) I_n(\lambda) d\lambda}{\int_0^\pi I_n(\lambda) d\lambda} \ , \tag{1.9}$$

only, the usual frequency domain bootstrap of Franke and Härdle works asymptotically. The reason is that the asymptotic distribution of a ratio statistics, which is again a normal distribution, does not depend on the fourth order cumulant of $\sigma\varepsilon_1$. In this case the asymptotic variance for ratio statistics is equal to

$$2\pi \int_0^\pi \psi^2 f^2 \ d\lambda / (\int f d\lambda)^4 \ , \tag{1.10}$$

where $\psi = \varphi \int f - \int \varphi f$ (cf. Dahlhaus and Janas (1996), p.1939). Nevertheless we think that the autoregressive aided frequency domain bootstrap presented in this paper also may outperform the finite sample behaviour of the standard frequency domain bootstrap for ratio statistics because the dependence structure of the periodogram ordinates is mimicked to a certain extend in our bootstrap proposal. The standard frequency domain bootstrap

of Franke and Härdle treats the periodogram ordinates as independent random variables, which they are only asymptotically.

To describe the basic idea behind our procedure recall first that under certain assumptions on the moment structure of the error process and the rate of decrease of the coefficients $\{\alpha_\nu\}$ in (1.1) (see assumptions (A1) and (A2) in Section 6), we have

$$E(I_n(\lambda_j)) = f(\lambda_j) + O(n^{-1}) \tag{1.11}$$

and

$$\mathrm{Cov}(I_n(\lambda_j), I_n(\lambda_k)) = \begin{cases} f^2(\lambda_j) + O(n^{-1}) & \text{for} \quad j = k \\ n^{-1} f(\lambda_j) f(\lambda_k) \Big(\frac{E\varepsilon_1^4}{\sigma^4} - 3 \Big) + o(n^{-1}) & \text{for} \quad j \neq k; \end{cases} \tag{1.12}$$

cf. Brillinger (1981).

Consider next the autoregressive process $\widetilde{\mathbf{X}} = \{\widetilde{X}_t : t \in \mathbb{Z}\}$ defined by

$$\widetilde{X}_t = \sum_{\nu=1}^{P} a_\nu(P) \widetilde{X}_{t-\nu} + \sigma(P)\ \tilde{\varepsilon}_t \tag{1.13}$$

where $\mathbf{a}(P) = (a_1(P), a_2(P), \ldots, a_P(P))' = \mathbf{\Gamma}(P)^{-1} \boldsymbol{\gamma}_P$, $\mathbf{\Gamma}(P) = (\gamma(i-j))_{i,j=1,2,\ldots,P}$, $\boldsymbol{\gamma}_P = (\gamma(1), \gamma(2), \ldots, \gamma(P))'$ and $\tilde{\varepsilon}_t$ is an i.i.d. sequence with mean zero and unit variance. Let $\sigma^2(P) = \gamma(0) - \mathbf{a}(P)' \mathbf{\Gamma}^{-1}(P) \mathbf{a}(P)$ and assume that $E(\tilde{\varepsilon}_t^4) < \infty$. Note that $\mathbf{\Gamma}(P)^{-1}$ exists for every $P \in I\!N$ provided $\gamma(0) > 0$ and $\gamma(h) \to 0$ as $h \to \infty$; cf. Brockwell and Davis (1991), Prop. 5.1.1. Furthermore, $\mathbf{a}(P)$ is the vector of coefficients of the best autoregressive fit in L_2-distance, i.e., the coefficients $(a_1(P), a_2(P), \ldots, a_P(P))'$ are defined uniquely as the argmin of the norm $E(X_t - \sum_{\nu=1}^{P} X_{t-\nu})^2$.

Let $f_{AR}(\lambda) = \sigma^2(P) |\Psi_{AR}(e^{-i\lambda})|^2$ be the spectral density of $\widetilde{\mathbf{X}}$, where $\Psi_{AR}(z) = 1/(1 - \sum_{\nu=1}^{P} a_\nu(P) z^\nu) =: 1/A_P(z)$, and consider random variables $Y_n(\lambda_j)$, $j = 0, 1, 2, \ldots, N$, defined by

$$Y_n(\lambda_j) = q(\lambda_j) \tilde{I}_n(\lambda_j) \tag{1.14}$$

where

$$q(\lambda) \ = \ \frac{f(\lambda)}{f_{AR}(\lambda)} \tag{1.15}$$

and $\tilde{I}_n(\lambda) = (2\pi n)^{-1} |\sum_{t=1}^{n} \widetilde{X}_t \exp\{-i\lambda t\}|^2$, i.e., $\tilde{I}_n(\lambda)$ is the periodogram based on observations $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ from $\widetilde{\mathbf{X}}$. Since the periodogram $\tilde{I}_n(\lambda)$ satisfies (1.11) and (1.12) with f replaced by f_{AR} and $E(\varepsilon_1^4)/\sigma^4$ replaced $E(\tilde{\varepsilon}_1)^4/\sigma^4(P)$ we get using (1.15) that

$$E(Y_n(\lambda_j)) = f(\lambda_j) + O(n^{-1}) \tag{1.16}$$

and

$$\mathrm{Cov}(Y_n(\lambda_j), Y_n(\lambda_k)) = \begin{cases} f^2(\lambda_j) + O(n^{-1}) & \text{for} \quad j = k \\ n^{-1} f(\lambda_j) f(\lambda_k) \Big(\frac{E\tilde{\varepsilon}_1^4}{\sigma^4(P)} - 3 \Big) + o(n^{-1}) & \text{for} \quad j \neq k. \end{cases} \tag{1.17}$$

Furthermore, and as for the ordinary periodogram ordinates, for a set of frequencies $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \pi$ the random vector $(Y_n(\lambda_1), Y_n(\lambda_2), \ldots, Y_n(\lambda_m))'$ is asymptotically distributed as a vector of independent and exponentially distributed variables, the sth component of which has mean $f(\lambda_s)$ and variance $f^2(\lambda_s)$.

Since the random variables $Y_n(\lambda_j)$ resample closely the random behavior of the periodogram ordinates $I_n(\lambda_j)$ the above results suggest the following: If the (asymptotic) distribution of a statistic based on $I_n(\lambda_j)$ is not affected by the dependence of the periodogram, then this distribution can be well approximated by the distribution of the corresponding statistic based on the random variables $Y_n(\lambda_j)$. Furthermore, if $E\tilde{\varepsilon}_1^4/\sigma^4(P)$ is close to $E\varepsilon_1^4/\sigma^4$ such an approximation will be also valid for periodogram statistics for which the dependence of the periodogram ordinates affects the asymptotic distribution of interest. We expect this to be true since in this case, and as equation (1.17) shows, the covariance of the random variables $Y_n(\lambda_j)$ mimics correctly also the covariance of the corresponding periodogram ordinates $I_n(\lambda_j)$. An important case where this is true is the case where the underlying process $\mathbf{X}$ belongs to the infinite order autoregressive class (1.4) and the order P of the autoregressive approximation increases (at an appropriate rate) as the sample size n increases.

An implementation of the above idea for bootstrapping the periodogram is presented in the next section. We mention here, however, that the basic idea underlying the bootstrap procedure proposed in this paper, which combines a parametric autoregressive approximation of the process $\mathbf{X}$ with a nonparametric 'correction' function q to resample the stochastic behavior of the periodogram $I_n(\lambda)$, can be also applied using parametric classes of processes other than the autoregressive one. For instance, we could also had considered a finite order moving average approximation of the process $\mathbf{X}$ and define, in a similar way as in (1.15), an appropriate correction function q. This will make a restriction to the process class (1.4) for statistics like those given in (1.5) superfluous. However, we rely in the following on the autoregressive approximation because it is a computationally easier and faster technique which is successful in many situations; cf. Berk (1974).

The paper is organized as follows. Section 2 describes in detail the proposed bootstrap procedure. Section 3 deals with applications of this procedure in approximating the sampling behavior of the so-called ratio statistics and of spectral means while Section 4 deals with nonparametric estimators in the frequency domain. In Section 5 some practical issues are discussed and a small simulation example is presented. The technical assumptions needed are stated in Section 6 while the proofs of the main theorems as well as of some technical lemmas are deferred to Section 7.

2. The Bootstrap Procedure

Based on the motivation given in the introduction, the bootstrap procedure, which is investigated in this paper can be described along the following five steps

I: Given the observations $X_1, ..., X_n$ we fit an autoregressive process of order P, where P may depend on the sample.

This leads to estimated parameters $\hat{a}_1(P), ..., \hat{a}_P(P)$ and $\hat{\sigma}(P)$, which are usually obtained from the common Yule-Walker equations; cf. Brockwell and Davis (1991). Consider the estimated residuals

$$\hat{\varepsilon}_t = X_t - \sum_{\nu=1}^{P} \hat{a}_\nu(P) X_{t-\nu} \ , \ t = P+1, ..., n \ ,$$

and denote by $\hat{F}_n^c$ the empirical distribution of the standardized quantities $\hat{\varepsilon}_{P+1}, ..., \hat{\varepsilon}_n$, i.e. $\hat{F}_n^c$ has mean zero and unit variance.

II: Generate bootstrap observations $X_1^+, X_2^+, \ldots, X_n^+$, according to the following autoregressive model of order P

$$X_t^+ = \sum_{\nu=1}^{P} \hat{a}_\nu(P) X_{t-\nu}^+ + \hat{\sigma}(P) \cdot \varepsilon_t^+ \ ,$$

where (ε_t^+) constitutes a sequence of i.i.d. random variables with cumulative distribution function $\hat{F}_n^c$ (conditionally on the given observations $X_1, ..., X_n$).
The bootstrap process $\mathbf{X}^+ = (X_t^+ : t \in \mathbb{Z})$ possesses the following spectral density

$$\hat{f}_{\mathrm{AR}}(\lambda) = \frac{\hat{\sigma}^2(P)}{2\pi} \left| 1 - \sum_{\nu=1}^{P} \hat{a}_\nu(P) e^{-i\nu\lambda} \right|^{-2} \ , \ \lambda \in [0, \pi] \ .$$

Note, that if we make use of the Yule-Walker parameter estimate in Step 1 then it is always ensured that $\hat{f}_{\mathrm{AR}}$ is well-defined, i.e. the polynomial $1 - \sum_{\nu=1}^{P} \hat{a}_\nu(P) z^\nu$ has no complex roots with magnitude less than or equal to one. Moreover, in the case of Yule Walker-estimates, the bootstrap autocovariances $\gamma^+(h) = E^+ X_1^+ X_{1+h}^+ \ , \ h = 0, 1, ..., P$ coincide with the empirical autocovariances $\hat{\gamma}_n(h)$ of the underlying observations.

III: Compute the periodogram of the bootstrap observations, i.e.

$$I_n^+(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t^+ e^{-i\lambda t} \right|^2 \ , \ \lambda \in [0, \pi] \ .$$

IV: Define the following nonparametric estimator $\hat{q}$

$$\hat{q}(\lambda) = \frac{1}{nh} \sum_{j=-N}^{N} K\left(\frac{\lambda - \lambda_j}{h} \right) \frac{I_n(\lambda_j)}{\hat{f}_{\mathrm{AR}}(\lambda_j)} \ , \ \lambda \in [0, \pi] \ .$$

Here, and above, the λ_j's denote the Fourier frequencies, $K : [-\pi, \pi] \to [0, \infty)$ denotes a probability density (kernel), and $h > 0$ is the so-called bandwidth.

V: Finally, the bootstrap periodogram I_n^* is defined as follows:

$$I_n^*(\lambda) = \hat{q}(\lambda) I_n^+(\lambda) \ , \ \lambda \in [0, \pi] \ .$$

Some remarks are now in order. Although the theory developed in the next sections allows for a data-dependent order P to be as flexible as possible a nonparametric correction in the final step is introduced in order to catch data-features, which can or are not represented by the autoregressive fit. This nonparametric correction is done via the function $\hat{q}$ and there are several reasons justifying its use. First of all the nonparametric correction in steps 4 and 5 makes the proposed bootstrap procedure applicable to a more general class of stochastic processes than the purely autoregressive bootstrap. As will be seen in the next sections, this is in particular true for periodogram statistics based on realizations of the process (1.1) which can not be captured by the purely autoregressive bootstrap, i.e., by the corresponding statistics based on the pseudo-periodogram values $I_n^+(\lambda)$. On the other hand, the parametric approximation makes the new procedure more general than the nonparametric Franke/Härdle bootstrap procedure. This is true since the new procedure leads to asymptotically valid approximations for a larger class of periodogram statistics than the Franke/Härdle procedure if the underlying process belongs to the important infinite order autoregressive class (1.4). Finally, and as in the spirit of the so-called pre-whitening idea in nonparametric spectral density estimation (cf. Press and Tukey (1956)), we expect an improved behavior of the spectral density estimator $\hat{q} \cdot \hat{f}_{AR}$ (implicitly) used by our bootstrap procedure. If, for example, the true underlying spectral density has some dominant peaks, then pre-whitening leads to a considerable improvement of the estimator. The reason is that an autoregressive fit is really able to catch the peaks of the spectral density rather well and the curve $I_n(\lambda)/\hat{f}_{\mathrm{AR}}(\lambda)$ is much smoother than $I_n(\lambda)$, thus much easier to estimate nonparametrically.

To elaborate on the differences between the nonparametric bootstrap procedure of Franke and Härdle (1992) and the autoregressive aided periodogram bootstrap proposed in this paper, recall first that under the assumptions of the paper and the definition of q, we have

$$I_n(\lambda_j) = 2\pi q(\lambda_j) f_{AR}(\lambda_j) I_{n,\varepsilon}(\lambda_j) + R_n(\lambda_j) \tag{2.1}$$

where $I_{n,\varepsilon}(\lambda)$ denotes the periodogram of the i.i.d. series $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ and the remainder $R_n(\lambda_j)$ satisfies $\max_j E(R_n(\lambda_j))^2 = O(n^{-1})$; cf. Brockwell and Davis (1991), Prop. 10.3.1. Furthermore, by Lemma 7.2(ii) and the fact that $I_n^*(\lambda) = \hat{q}(\lambda) I_n^+(\lambda)$, a similar expression can be obtained for $I_n^*(\lambda_j)$, i.e., we have

$$I_n^*(\lambda_j) = 2\pi \hat{q}(\lambda_j) \hat{f}_{AR}(\lambda_j) I_{n,\varepsilon^*}(\lambda_j) + R_n^*(\lambda_j), \tag{2.2}$$

where $I_{n,\varepsilon^*}(\lambda)$ denotes the periodogram of the i.i.d. series $\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_n^*$ and the remainder $R_n^*(\lambda_j)$ satisfies $\max_j E^* R_n^2(\lambda_j) = O_P(n^{-1})$; cf. lemma 7.3. Note that the $I_{n,\varepsilon^*}(\lambda_j)$ are not independent. Finally, recall that the bootstrap periodogram ordinates of the Franke/Härdle bootstrap procedure are given by $I_n^\star(\lambda_j) = \hat{f}_h(\lambda_j) U_j^\star$ where $\hat{f}_h(\lambda) = (nh)^{-1} \sum_{j=-N}^{N} K((\lambda - \lambda_j)/h) I_n(\lambda_j)$ and $U_j^\star$ is an i.i.d. sequence based on the rescaled residuals $\hat{U}_j = I_n(\lambda_j)/\hat{f}(\lambda_j)$. Thus, the Franke/Härdle bootstrap differs from the autoregressive aided bootstrap not only by the independence of the generated bootstrap periodogram ordinates $I_n^\star(\lambda_j)$ but also by the estimator of the spectral density f used. In particular, in the autoregressive aided periodogram bootstrap the kernel estimator $\hat{f}_h$ is replaced by the (implicitly used) autoregressive aided spectral density estimator $\tilde{f} = \hat{q} \cdot \hat{f}_{AR}$. Note

that equations (7.27) and (7.28) imply that for $P \in N$ fixed this estimator is uniformly consistent, i.e.,

$$\sup_{\lambda \in [0,\pi]} |\widetilde{f}(\lambda) - f(\lambda)| \ \to \ 0$$

in probability.

3. Spectral Means and Ratio Statistics

The bootstrap analog of (1.6) now reads as follows

$$\sqrt{n} \left(\int_0^\pi \varphi(\lambda) I_n^*(\lambda) d\lambda - \int_0^\pi \varphi(\lambda) \hat{q}(\lambda) \hat{f}_{\rm AR}(\lambda) d\lambda \right) \ . \tag{3.1}$$

Alternatively, we may, as above, consider the following discretized bootstrap statistic

$$\frac{2\pi}{\sqrt{n}} \sum_{j=0}^{N} \varphi(\lambda_j) \left(I_n^*(\lambda_j) - \hat{q}(\lambda_j) \hat{f}_{\rm AR}(\lambda_j) \right) d\lambda \ . \tag{3.2}$$

We can now show the following theorem, which states that our bootstrap procedure works.

Theorem 3.1 **(i)** *Assume (B1)-(B2) and (A2)-(A7). Then we have (in probability)*

$$\begin{aligned} \mathcal{L} \left[\sqrt{n} \left(\int_0^\pi \varphi(\lambda) I_n^*(\lambda) d\lambda - \int_0^\pi \varphi(\lambda) \hat{q}(\lambda) \hat{f}_{\rm AR}(\lambda) d\lambda \right) \Big| X_1, ..., X_n \right] \\ \Rightarrow \mathcal{N} \left(0, 2\pi \int_0^\pi \varphi^2 f^2 d\lambda + \kappa_4 \left(\int_0^\pi \varphi f d\lambda \right)^2 \right) \end{aligned} \tag{3.3}$$

(ii) *Assume (A1)-(A7). Then we have for all fixed $P \in I\!N$ that the same assertion as in (i) holds true with κ_4 replaced by $\kappa_4(P) = E \left(X_P - \sum_{\nu=1}^P a_\nu(P) X_{P-\nu} \right)^4 / \sigma(P)^4 - 3$.*

From Dahlhaus and Janas (1996) it is known that the standard frequency domain bootstrap works for the so-called ratio statistics (cf. (1.9)). Thus it is worth to study the behaviour of our autoregressive aided frequency domain bootstrap for such statistics. From Theorem 3.1(i) it is clear that under the assumptions of this part of the theorem our bootstrap proposal works for ratio statistics. More interesting is the question whether the autoregressive aided frequency domain bootstrap works for ratio statistics even if we keep the order P of the autoregressive fit fixed. To this end observe that we have as in Dahlhaus and Janas (1996)

$$\begin{aligned} & \sqrt{n} \left(\frac{\int_0^\pi \varphi(\lambda) I_n^*(\lambda) d\lambda}{\int_0^\pi I_n^* d\lambda} - \frac{\int_0^\pi \varphi(\lambda) \hat{q}(\lambda) \hat{f}_{\rm AR}(\lambda) d\lambda}{\int_0^\pi \hat{q}(\lambda) \hat{f}_{\rm AR}(\lambda) d\lambda} \right) \\ = \ & \frac{\sqrt{n}}{\int \hat{q} \hat{f}_{\rm AR} d\lambda \int I_n^* d\lambda} \int_0^\pi \widetilde{\psi}(\lambda) I_n^*(\lambda) d\lambda \ , \end{aligned} \tag{3.4}$$

with $\widetilde{\psi}(\lambda) = \varphi(\lambda) \int \hat{q} \hat{f}_{\rm AR} d\lambda - \int \varphi \hat{q} \hat{f}_{\rm AR} d\lambda$. Since $\int \widetilde{\psi} \hat{q} \hat{f}_{\rm AR} d\lambda = 0$ (which implies that also the limit $\int \psi f \ d\lambda$ is equal to zero and therefore from Theorem 3.1 (i) that the asymptotic distribution does not depend on fourth order cumulants) we immediately obtain the following corollary.

Corollary 3.1 **(i)** *Assume (B1)-(B2) and (A2)-(A6). Then we have (in probability)*

$$\mathcal{L}\left[\sqrt{n}\left(\frac{\int_0^\pi \varphi(\lambda)I_n^*(\lambda)d\lambda}{\int_0^\pi I_n^*(\lambda)d\lambda} - \frac{\int_0^\pi \varphi(\lambda)\hat{q}(\lambda)\hat{f}_{\mathrm{AR}}(\lambda)d\lambda}{\int_0^\pi \hat{q}(\lambda)\hat{f}_{\mathrm{AR}}(\lambda)d\lambda}\right)\Big| X_1,...,X_n\right] \tag{3.5}$$
$$\Rightarrow \mathcal{N}\left(0, 2\pi\int_0^\pi \psi^2(\lambda)f^2(\lambda)d\lambda/(\int f d\lambda)^4\right) ,$$

where $\psi(\lambda) = \varphi(\lambda)\int f(u)\ du - \int \varphi(u)f(u)\ du$.

(ii) *Assume (A1)-(A6). Then we have for all fixed* $P \in \mathbb{N}$ *that the same assertion as in (i) holds true.*

Thus, in both cases the limiting normal distribution in the bootstrap world coincides with the limiting distribution of ratio statistics (cf. (1.10)).

4. Nonparametric Estimators

An interesting class of spectral density estimators is given by

$$\hat{f}(\lambda) = \frac{1}{nb}\sum_{j=-N}^{N} K\Big(\frac{\lambda-\lambda_j}{b}\Big)I_n(\lambda_j) \tag{4.1}$$

where $K(\cdot)$ is the kernel and $b = b(n)$ the bandwidth. In the following we are interested in estimating the distribution of the statistic

$$\sqrt{nb}\ (\hat{f}(\lambda) - f(\lambda)). \tag{4.2}$$

For this, the bootstrap statistic

$$\sqrt{nb}\ (\hat{f}^*(\lambda) - \widetilde{f}(\lambda)) \tag{4.3}$$

can be used, where

$$\hat{f}^*(\lambda) = \frac{1}{nb}\sum_{j=-N}^{N} K\Big(\frac{\lambda-\lambda_j}{b}\Big)I_n^*(\lambda_j) \tag{4.4}$$

and

$$\widetilde{f}(\lambda) = \hat{q}(\lambda)\hat{f}_{AR}(\lambda). \tag{4.5}$$

The following theorem shows that if the underlying process satisfies (1.1) then the proposed bootstrap procedure works. For this and in order to metrize the distance between distributions, we use in the following theorem Mallow's d_2 metric on the space $\{\mathcal{P} : \mathcal{P}$ probability measure on $(\mathbb{R}, \mathcal{B}), \int |x|^2 d\mathcal{P} < \infty\}$. This metric is defined according to $d_2(\mathcal{P}_1, \mathcal{P}_2) = \inf\{E|Y_1 - Y_2|^2\}^{1/2}$ where the infimum is taken over all real-valued random variables (Y_1, Y_2) which have marginal distributions $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively; cf. Bickel and Freedman (1981) for more details.

Theorem 4.1 *Suppose that assumptions (A1)-(A6) and (A8)-(A9) are satisfied. Then we have for all fixed $P \in \mathbb{N}$ that*

(i) *If $nb^5 \to 0$ then*

$$d_2\Big\{\mathcal{L}(\sqrt{nb}\ (\hat{f}(\lambda) - f(\lambda))), \mathcal{L}(\sqrt{nb}\ (\hat{f}^*(\lambda) - \widetilde{f}(\lambda))|X_1, X_2, \ldots, X_n)\Big\} \to 0$$

in probability.

(ii) *If $b \sim n^{-1/5}$ and $nh^3 \to \infty$ then the same result as in (i) holds true.*

To elaborate on the assumption $nh^3 \to \infty$ needed in the second part of the above theorem note that if $nb^5 \to 1$ then the bias $E(\hat{f}(\lambda)) - f(\lambda)$ of the nonparametric estimator (4.1) is asymptotically not negligible. It rather converges to $(1/4\pi)f''(\lambda)\int u^2K(u)du$ as $n \to \infty$; cf. Priestley (1981). In order to provide a valid approximation of the distribution of $\sqrt{nb}(\hat{f}(\lambda) - f(\lambda))$ in this case, too, the bootstrap has to be able to estimate the bias term correctly. The condition $nh^3 \to \infty$ implies that h tends to zero slower than b, i.e., $\hat{q}(\lambda)$ should be somewhat smoother than the optimal (with respect to minimizing the mean square error) kernel estimator of q. Therefore, the above assumption can be interpreted as an over-smoothing assumption, which is common in applications of the bootstrap to approximate the bias in nonparametric estimation; cf. Romano (1988), Franke and Härdle (1992), and Paparoditis and Politis (1999).

5. Practical Aspects and Numerical Examples

In this section we study and compare the performance of the proposed autoregressive aided periodogram bootstrap, that of the autoregressive bootstrap and of the nonparametric periodogram bootstrap proposed by Franke and Härdle (1992).

In order to make such a comparison we choose in the following a statistic for which all three methods lead to asymptotically correct approximations but no one of them is expected to have a particular advantage. To be more specific we study and compare the performance of the three aforementioned bootstrap methods in estimating the standard deviation $\hat{\sigma}_1$ of the first order sample autocorrelation $\hat{\rho}_n(1) = \hat{\gamma}_n(1)/\hat{\gamma}_n(0)$ where $\hat{\gamma}_n(h) = n^{-1}\sum_{t=1}^{n-h}(X_t - \overline{X})(X_{t+h} - \overline{X})$ is the sample autocovariance at lag h and $\overline{X} = n^{-1}\sum_{t=1}^{n} X_t$. Realizations of length $n = 50$ from the model

$$X_t = \varepsilon_t + \theta\varepsilon_{t-1}$$

with $\theta = 0.95$ and $\varepsilon_t \sim N(0,1)$ have been considered.

To estimate the exact standard deviation of $\hat{\rho}_n(1)$, 1000 replications have been used while the bootstrap approximations are based on $B = 300$ bootstrap replications. All three methods have been applied by using nonrandom as well as data driven choices of the bootstrap parameters. In particular, the performance of the autoregressive based and

autoregressive aided method has been studied for three different choices of the autoregressive order P and a choice of P based on the AIC criterion, i.e., the order P_{AIC} which minimizes the function $\mathrm{AIC}(P) = \mathrm{argmin}_P\{\hat{\sigma}^2(P) \cdot (1 + 2P/n)\}$ over a range of values $P = 1, 2, \ldots, P_{max}$, where $P_{max} = 10 \cdot \log_{10}(n)$. Similarly, the nonparametric frequency domain bootstrap has been applied using three different choices of the smoothing parameter h and a choice of h based on the cross validation criterion of Beltrão and Bloomfield (1987). Recall that in the Franke/Härdle bootstrap, h is the bandwidth used to obtain $\hat{f}_h$ which is a kernel estimator of f. According to this criterion we select h as the minimizer of the function

$$\mathrm{CV}(h) = \frac{1}{N} \sum_{j=1}^{N} \left\{ \log \hat{f}_{-j}(\lambda_j) + \frac{I_n(\lambda_j)}{\hat{f}_{-j}(\lambda_j)} \right\} \tag{5.1}$$

where

$$\hat{f}_{-j}(\lambda_j) = \frac{1}{nh} \sum_{j \in \mathbf{N}_j} K\left(\frac{\lambda_j - \lambda_s}{h} \right) I_n(\lambda_j) \tag{5.2}$$

and $\mathbf{N}_j = \{s : -N \leq s \leq N \text{ and } j - s \neq \pm j \mathrm{mod} N\}$. That is, $\hat{f}_{-j}$ is the kernel estimator of f when the jth periodogram ordinate is deleted. This cross validation criterion has been also applied to choose the smoothing bandwidth h used to estimate the nonparametric correction function $q(\cdot)$ of the autoregressive aided periodogram bootstrap. In this case the periodogram $I_n(\lambda)$ appearing in (5.1) and (5.2) has been replaced by $I_n(\lambda)/\hat{f}_{AR}(\lambda)$.

The results are summarized in Table 1 where the mean value, the standard deviation and the mean square error of the three bootstrap approximations are reported as sample moments over 200 simulations.

Please insert Table 1 about here

As the entries of Table 1 show, the results of the autoregressive aided periodogram bootstrap compare favorable with those of the other two methods. In particular and compared to the nonparametric bootstrap we observe an overall decrease in the mean square error of the new bootstrap estimator. The table shows also a decrease in the variability of the mean square error of the autoregressive aided periodogram bootstrap compared to that of the purely autoregressive bootstrap over the different choices of the bootstrap parameters, i.e., the autoregressive order P. As this table confirms, this decrease is mainly due to a reduction of the bias of the bootstrap estimator which is caused by the nonparametric correction applied. The results based on the new bootstrap procedure seem also to be less sensitive with respect to the choice of the corresponding bootstrap parameters, which is probably due to the frequency domain nonparametric correction via the function $\hat{q}$. The effect of this nonparametric correction is clearly seen in comparing the results of the autoregressive aided periodogram bootstrap to those of the purely autoregressive bootstrap for the case $P = 1$.

6. Assumptions

(A1) $\{X_t; t \in \mathbb{Z}\}$ is a real-valued process

$$X_t = \sigma \cdot \sum_{\nu=0}^{\infty} \alpha_\nu \varepsilon_{t-\nu}, \qquad \alpha_0 = 1,$$

where $\sum_{\nu=0}^{\infty} \nu^2 |\alpha_\nu| < \infty$ and $\{\varepsilon_t; t \in \mathbb{Z}\}$ are independent and identically distributed random variables.

(A2) $E\varepsilon_t = 0$, $E\varepsilon_t^2 = 1$ and $E\varepsilon_t^4 < \infty$. Furthermore, $\sigma \in (0, \infty)$ and κ_4 denotes the fourth order cumulant of ε_t .

(A3) The spectral density f of $\{X_t\}$ is Lipschitz continuous and satisfies $\inf_{\lambda \in [0,\pi]} f(\lambda) > 0$.

(A4) $K(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} k(u) e^{-i\lambda u} du$ where $k(u)$ is a nonnegative, continuous and even function with $k(0) = 1$, $k(u) = 0$ for $|u| > 1$ and $\int_{-\infty}^{\infty} k^2(x) dx < \infty$.

(A5) $h \to 0$ as $n \to \infty$ such that $nh \to \infty$.

(A6) $b \to 0$ as $n \to \infty$ such that $nb \to \infty$.

(A7) $\varphi : [0, \pi] \to I\!R$ is a bounded function having bounded variation.

(A8) The spectral density f of $\mathbf{X}$ is three times continuous differentiable on $[-\pi, \pi]$.

(A9) K is three times continuously differentiable on $[-\pi, \pi]$.

(B1) $\{X_t; t \in \mathbb{Z}\}$ is a real-valued process

$$X_t = \sum_{\nu=1}^{\infty} a_\nu X_{t-\nu} + \sigma \varepsilon_t,$$

where $\sum_{\nu=1}^{\infty} \sqrt{\nu} |a_\nu| < \infty$ and $1 - \sum_{\nu=1}^{\infty} a_\nu z^\nu \neq 0$ for all complex z with $|z| \leq 1$.

(B2) $P(n) \in [p_{\min}(n), p_{\max}(n)]$ where $p_{\min}(n) \to \infty$ as $n \to \infty$ and $\frac{p_{\max}(n)^5 \log n}{n}$ stays bounded. Observe that this means that $P(n)$ may be chosen data-dependent.

7. Proofs and Technical Lemmas

Let us collect some properties of the process $\mathbf{X}^+$ under the assumptions (B1) and (B2). For all n large enough the process $\mathbf{X}^+$ possesses the following moving average representation

$$X_t^+ = \hat{\sigma}(P) \sum_{\nu=0}^{\infty} \hat{\alpha}_\nu(P) \varepsilon_{t-\nu}^+ \ , \ t \in \mathbb{Z} \ .$$

The coefficients $(\hat{\alpha}_\nu(P) : \nu \in I\!N_o)$ can be computed as follows ($\hat{\alpha}_0(P) = 1$):

$$\left(1 - \sum_{\nu=1}^{P} \hat{a}_\nu(P) z^\nu\right)^{-1} = 1 + \sum_{\nu=1}^{\infty} \hat{\alpha}_\nu(P) z^\nu \text{ for all } |z| \leq 1 \ . \tag{7.1}$$

From Lemma 8.3, Kreiss (1999), we have that uniformly in $\nu \in I\!N$

$$|\hat{\alpha}_\nu(P) - \alpha_\nu(P)| \leq \frac{P}{(1+1/P)^\nu} \mathcal{O}_P(\sqrt{\log n/n}) \ , \tag{7.2}$$

where $(\alpha_\nu(P) : \nu \in I\!N)$ is exactly defined as $(\hat{\alpha}_\nu(P) : \nu \in I\!N)$, cf. (7.1), with $\hat{a}_\nu(P)$ replaced by $a_\nu(P), \nu = 1, ..., P$.

Furthermore we have

$$\sum_{\nu=0}^{\infty} |\alpha_\nu(P) - \alpha_\nu| \leq \mathcal{O}_P(\sum_{\nu=P}^{\infty} |a_\nu|) = o_P(1) \ . \tag{7.3}$$

Finally this implies for the autocovariances of the process $\mathbf{X}^+$:

$$\gamma^+(k) = E^+ X_t^+ X_{t+k}^+ = \sum_{\nu=0}^{\infty} \hat{\alpha}_{\nu+k}(P) \hat{\alpha}_\nu(P) \hat{\sigma}^2(P) \ , \ k \in I\!N_o \ , \tag{7.4}$$

and

$$\sum_{k=0}^{n} |\gamma^+(k)| = \mathcal{O}_P(1) \ . \tag{7.5}$$

This absolute summability of the autocovariances of the process $\mathbf{X}^+$ implies (by Kronecker's lemma, cf. Bauer (1974), Lemma 61.1)

$$\frac{1}{n} \sum_{k=0}^{n} k|\gamma^*(k)| = o_P(1) \ . \tag{7.6}$$

In case we do not have an autoregressive representation of the underlying process $\mathbf{X}$, but only the infinite order moving average representation, i.e. (A1) and (A2) hold true, and if we additionally assume that P is a fixed integer, then the behaviour of the bootstrap process $\mathbf{X}^+$ is much easier. It is well-known that the autocorrelations $\hat{\varrho}(k) = \hat{\gamma}(k)/\hat{\gamma}(0) \ , k = 1, ..., P$ are $\sqrt{n}$-consistent. This implies that the autoregressive coefficients $\hat{a}_1(P), ..., \hat{a}_P(P)$ are $\sqrt{n}$-consistent for $a_1(P), ..., a_P(P)$. Further, it is always true that $1 - \sum_\nu \hat{a}_\nu(P) z^\nu$ as well as $1 - \sum_\nu a_\nu(P) z^\nu$ have no complex roots with magnitude less than or equal to one, cf. Brockwell and Davis (1991), p. 240. Thus in this case the bound in (7.2) is just $\mathcal{O}_P(1/\sqrt{n})$. Of course (7.3) does not hold true, because P is fixed, but (7.4)-(7.6) are valid as stated.

Lemma 7.1 (**i**) *Assume (B1), (B2) and (A2), f, f^{-1} and f' uniformly bounded on $[0, \pi]$, $h \to 0$ and $nh \to \infty$. Then we have for all $\lambda \in [0, \pi]$*

$$\hat{q}(\lambda) \to 1 \text{ in probability} \ .$$

Moreover

$$\int_0^\pi |\hat{q}(\lambda) - 1|\, d\lambda \to 0 \text{ in probability } .$$

(**ii**) *Assume (A1) and (A2), f, f^{-1} and f' uniformly bounded on $[0,\pi]$, $h \to 0$ and $nh \to \infty$. Then we have for all $P \in \mathbb{N}$ (fixed) and all $\lambda \in [0,\pi]$*

$$\hat{q}(\lambda) \to q(\lambda) \text{ in probability}$$

and

$$\int_0^\pi |\hat{q}(\lambda) - q(\lambda)|\, d\lambda \to 0 \text{ in probability } ,$$

where $q(\lambda) = f(\lambda)/f_{\mathrm{AR}}(\lambda)$, cf. (1.15).

Proof: Since the arguments are rather similar to those given for the consistency of the smoothed periodogram we only give a sketch for the more complicated part (i).

From Kreiss (1999), especially formulas (8.4) and (8.8), we obtain

$$\sup_{\lambda\in[0,\pi]} \left|\hat{f}_{\mathrm{AR}}^{-1}(\lambda) - f^{-1}(\lambda)\right| \;=\; o_P(1) \ .$$

Here we make use of the representation

$$\hat{f}_{\mathrm{AR}}^{-1}(\lambda) = \frac{2\pi}{\sigma^2}\left|1 - \sum_{\nu=1}^{P} \hat{a}_\nu(P) e^{-i\nu\lambda}\right|^2 \ .$$

Together with the boundedness of f' and f^{-1} this leads to

$$\hat{q}_n(\lambda) = \frac{1}{nh}\sum_{j=-N}^{N} K\left(\frac{\lambda - \lambda_j}{h}\right) I_n(\lambda_j)/f(\lambda) + o_P(1) \ ,$$

where the vanishing remainder term does not depend on λ.

Finally it is well-known, that under our assumptions on the spectral density f and the bandwidth h it holds that for all $\lambda \in [0,\pi]$

$$\frac{1}{nh}\sum_{j=-N}^{N} K\left(\frac{\lambda - \lambda_j}{h}\right) I_n(\lambda_j) \to_{n\to\infty} f(\lambda)$$

in probability. Moreover, under more restrictive assumptions, a uniform convergence in λ is available, but this is not required for our purpose.

To see the second assertion of (i) it suffices to show that

$$\int_0^\pi \frac{1}{nh}\left|\sum_{j=-N}^{N} K\left(\frac{\lambda - \lambda_j}{h}\right)\left(I_n(\lambda_j) - EI_n(\lambda_j)\right)\right| \, d\lambda$$

converges to zero in probability, because $EI_n(\lambda)$ converges under our assumptions uniformly to $f(\lambda)$, cf. Brockwell and Davis (1991), Prop. 10.3.1. The variance of this last expression is bounded by (use Brockwell and Davis (1991), Theorem 10.3.2)

$$\begin{aligned} & \int_0^\pi \frac{1}{n^2h^2} \sum_{j=-N}^{N} K^2\left(\frac{\lambda-\lambda_j}{h}\right) \mathrm{Var} I_n(\lambda_j)\ d\lambda \\ + & \int_0^\pi \frac{1}{n^2h^2} \sum_{j,l=-N, l\neq j}^{N} K\left(\frac{\lambda-\lambda_j}{h}\right) K\left(\frac{\lambda-\lambda_l}{h}\right) \mathrm{Cov}(I_n(\lambda_j), I_n(\lambda_l))\ d\lambda \\ = & \frac{1}{n^2h^2}\mathcal{O}(nh) + \frac{1}{n^2h^2}\mathcal{O}(nh^2) = o(1)\ . \end{aligned}$$

■

Lemma 7.2 (**i**) *Assume (B1) and (B2). Then we have* $\sum_{k=0}^{\infty} \sqrt{k}|\hat{\alpha}_k(P)| < \infty$.

(**ii**) *Assume (A1) and (A2) and let* $P \in I\!N$ *be fixed. Let* $\tilde{\mathbf{a}} = (\tilde{a}_1(P), \tilde{a}_2(P), \ldots, \tilde{a}_P(P))'$ *be a* $\sqrt{n}$*-consistent estimator of* $\mathbf{a}(P)$ *which satisfies* $1 - \sum_{\nu=1}^{P} \tilde{a}_\nu(P) z^\nu \neq 0$ *for* $|z| \leq 1$. *Then*

$$\sum_{k=1}^{\infty} k^\delta |\tilde{\alpha}_k(P)| = O_P(1)$$

for any $\delta \in (0, \infty)$.

Proof: From Kreiss (1999), Lemma 8.3, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} \sqrt{k}|\hat{\alpha}_k(P) - \alpha_k(P)| & \leq \sum_{k=0}^{\infty} \sqrt{k}(1+1/P)^{-\nu} \mathcal{O}_P(\sqrt{P^2/n\log n}) \\ & \leq \mathcal{O}_P(\sqrt{P^5/n\log n}) = \mathcal{O}_P(1)\ . \end{aligned}$$

Thus, it suffices to consider

$$\sum_{k=0}^{\infty} \sqrt{k}|\alpha_k(P) - \alpha_k| \quad \text{and} \quad \sum_{k=0}^{\infty} \sqrt{k}|\alpha_k|\ .$$

$\sum_{k=0}^{\infty} \sqrt{k}|\alpha_k| < \infty$ follows from (B1) and Hannan and Kavalieris (1986). In order to see that the first series is convergent we obtain exactly along the lines of the proof of Lemma 8.2 in Kreiss (1999) that for all n large enough, i.e. P large enough

$$\sum_{k=0}^{\infty} \sqrt{k}|\alpha_k(P) - \alpha_k| \leq C \cdot \sum_{k=0}^{\infty} \sqrt{k}|a_k(P) - a_k|\ .$$

From Baxter (1962) we finally obtain that, again for all P large enough:

$$\sum_{k=0}^{\infty} \sqrt{k}|a_k(P) - a_k| \leq C' \cdot \sum_{k=P}^{\infty} \sqrt{k}|a_k| = o_P(1)\ ,$$

since $\sum_{k=0}^{\infty}\sqrt{k}|a_k| < \infty$. This implies (i).
To see (ii) recall that $1 - \sum_{k=1}^{P} a_k(P)z^k \neq 0$ for $z \leq 1$. Furthermore, for P fixed, $\epsilon > 0$ exists such that the power series $(1 - \sum_{k=1}^{P} a_k(P)z^k)^{-1} = 1 + \sum_{k=1}^{\infty}\alpha_k(P)z^k$ converges for $|z| < 1 + \epsilon$. This implies that $\alpha_k(P)(1+\epsilon/2)^k \to 0$ as $k \to \infty$, i.e., there exist positive constants $C > 0$ and $\rho \in (0,1)$ such that $|\alpha_k(P)| \leq C\rho^k$ for $k = 1, 2, \ldots$. Now,

$$\sum_{k=1}^{\infty} k^{\delta}|\widetilde{\alpha}_k(P)| \leq \sum_{k=1}^{\infty} k^{\delta}|\alpha_k(P)| + \sum_{k=1}^{\infty} k^{\delta}|\widetilde{\alpha}_k(P) - \alpha_k(P)|.$$

Using Lemma 2.2 of Kreiss and Franke (1992) to bound the difference $|\widetilde{\alpha}_k(P) - \alpha_k(P)|$ we get that for some constant $\eta > 0$,

$$\begin{aligned}\sum_{k=1}^{\infty} k^{\delta}|\alpha_k(P)| + \sum_{k=1}^{\infty} k^{\delta}|\widetilde{\alpha}_k(P) - \alpha_k(P)| &\leq O(1) + \sum_{k=1}^{\infty} k^{\delta}(1+\eta)^{-k}O_P(n^{-1/2}) \\ &= O_P(1).\end{aligned}$$

Note that the last equality follows using the fact that the $O_P(n^{-1/2})$ term is uniformly in k. ∎

Proof of Theorem 3.1: We prove part (i), only. First of all we show that we can restrict our considerations to

$$\sqrt{n}\int_0^{\pi} \varphi(\lambda)\left(I_n^+(\lambda) - E^+I_n^+(\lambda)\right)\hat{q}(\lambda)\, d\lambda\ . \tag{7.7}$$

To see this, observe that $I_n^*(\lambda) = I_n^+(\lambda)\hat{q}(\lambda)$ and that for all $\lambda \in [0,\pi]$

$$I_n^+(\lambda) \;=\; \frac{1}{2\pi n}\left|\sum_{t=1}^{n} X_t^+ e^{-i\lambda t}\right|^2 = \frac{1}{2\pi}\left\{\hat{\gamma}_n^+(0) + 2\cdot\sum_{k=1}^{n-1}\hat{\gamma}_n^+(k)\cos(\lambda k)\right\}\ ,$$

where $\hat{\gamma}_n^+(k) = \frac{1}{n}\sum_{t=1}^{n-k} X_t^+ X_{t+k}^+, k = 0, 1, 2, ..., n-1$.
We have $E^+\hat{\gamma}_n^+(k) = (1 - \frac{k}{n})E^+X_1^+X_{1+k}^+ = (1-\frac{k}{n})\gamma^+(k), k = 0, 1, ..., n-1$, $\hat{f}_{\mathrm{AR}}(\lambda) = \frac{\hat{\sigma}^2(P)}{2\pi}|1 - \sum_{\nu=1}^{P}\hat{a}_\nu(P)e^{-i\nu\lambda}|^{-2}$ and

$$\frac{\hat{\sigma}^2(P)}{2\pi}\left|1 - \sum_{\nu=1}^{P}\hat{a}_\nu(P)e^{-i\nu\lambda}\right|^{-2} = \frac{1}{2\pi}\left\{\gamma^+(0) + 2\cdot\sum_{k=1}^{\infty}\gamma^+(k)\cos(\lambda k)\right\}\ .$$

Thus, in order to see that the difference of expression (3.1) and (7.7) is $o_P(1)$ it suffices to show (7.8) and (7.9):

$$\sqrt{n}\sum_{k=n}^{\infty}\gamma^+(k)\int_o^{\pi}\varphi(\lambda)\hat{q}(\lambda)\cos(\lambda k)\, d\lambda = o_P(1) \tag{7.8}$$

$$\sum_{k=1}^{n-1}\frac{k}{\sqrt{n}}\gamma^+(k)\int_o^{\pi}\varphi(\lambda)\hat{q}(\lambda)\cos(\lambda k)\, d\lambda = o_P(1) \tag{7.9}$$

Boundedness of $\sum_{k=1}^{n}\sqrt{k}|\gamma^+(k)|$ in probability, Lemma 7.1 and the absolute summability of the Fourier-coefficients of φ imply (7.8) and (7.9).
(7.7) can be rewritten as

$$\begin{aligned} Z_n \;:=\; & \frac{1}{2\pi}\left(\sqrt{n}(\hat\gamma_n^+(k)-E^+\hat\gamma_n^+(k)) : k=0,1,...,n-1\right)\cdot \\ & \left(\int_o^\pi \varphi(\lambda)\hat q(\lambda)\; d\lambda, 2\int_0^\pi \varphi(\lambda)\hat q(\lambda)\cos(\lambda k)\; d\lambda : k=1,...,n-1\right)' + o_P(1)\;. \end{aligned} \tag{7.10}$$

From Kreiss (1999), Theorem 3.1, for each fixed $K\in\mathbb{N}$, we have that

$$\mathcal{L}\left(\sqrt{n}(\hat\gamma_n^+(k)-E^+\gamma_n^+(k)) : k=0,...,K|X_1,...,X_n\right)\Rightarrow\mathcal{N}(0,V_K)\;, \tag{7.11}$$

where

$$V_K=\left[\left(E\varepsilon_1^4-3\right)\gamma(i)\gamma(j)+\sum_{k=-\infty}^{\infty}(\gamma(k)\gamma(k-i+j)+\gamma(k+j)\gamma(k-i))\right]_{i,j=0}^{K}\;.$$

Because of Lemma 7.1 we can replace $\hat q(\cdot)$ in (7.10) by its limit 1. Denote by Z_n' the quantity which is defined as Z_n, cf. (7.10), with this replacement. For $K\in\mathbb{N}$ decompose Z_n' into the following two quantities.

$$\begin{aligned} Z_{n,K}' \;:=\; & \frac{1}{2\pi}\left(\sqrt{n}(\hat\gamma_n^+(k)-E^+\hat\gamma_n^+(k)) : k=0,1,...,K\right)\cdot \\ & \left(\int_o^\pi \varphi(\lambda)\; d\lambda, 2\int_o^\pi \varphi(\lambda)\cos(\lambda k)\; d\lambda : k=1,...,K\right)'\;. \end{aligned} \tag{7.12}$$

and

$$\begin{aligned} Z_n'-Z_{n,K}' \;:=\; & \frac{1}{\pi}\left(\sqrt{n}(\hat\gamma_n^+(k)-E^+\hat\gamma_n^+(k)) : k=K+1,...,n-1\right)\cdot \\ & \left(\int_0^\pi \varphi(\lambda)\cos(\lambda k)\; d\lambda : k=K+1,...,n-1\right)^T\;. \end{aligned} \tag{7.13}$$

In order to obtain the asymptotic normality stated in the theorem we have to show (cf. Brockwell and Davis (1991), Prop. 6.3.9):

$$Z_{n,K}'\Rightarrow\mathcal{N}(0,\tau_K^2)\qquad\text{for all}\;\; K\in\mathbb{N} \tag{7.14}$$

$$\tau_K^2\to\tau^2\qquad\text{as}\;\; K\to\infty \tag{7.15}$$

and

$$\lim_{K\to\infty}\limsup_{n\to\infty}P^+\{|Z_n'-Z_{n,K}'|>\varepsilon\}=0\qquad\text{for all}\;\;\varepsilon>0. \tag{7.16}$$

(7.14) is a direct consequence of (7.11). (7.16) can be seen as follows:

$$E^+|Z_n'-Z_{n,K}'|\le\sum_{k=K+1}^{n-1}\left(n\mathrm{Var}^+(\hat\gamma_n^+(k))\right)^{1/2}\cdot\left|\int_0^\pi\varphi(\lambda)\cos(\lambda k)\; d\lambda\right|\;.$$

Since $n\mathrm{Var}^+(\hat{\gamma}_n^+(k))$ is bounded (in probability) uniformly in k we obtain $\limsup_{n\to\infty} E^+|Z_n' - Z_{n,K}'| = o_P(1)$, as $K \to \infty$, because of the absolute summability of the Fourier-coefficients of φ.

It remains to show (7.15). In a first step it is easy to see that

$$\begin{aligned}\lim_{K\to\infty} \tau_K^2 &= \frac{1}{4\pi^2}\kappa_4 \left(\sum_{r=0}^{\infty} \hat{\varphi}_r \gamma(r)\right)^2 \\ &+ \frac{1}{4\pi^2} \sum_{k=-\infty}^{\infty} \sum_{r,s=0}^{\infty} \hat{\varphi}_r \hat{\varphi}_s \{\gamma(k)\gamma(k-r+s) + \gamma(k+s)\gamma(k-r)\} ,\end{aligned}$$

where

$$\kappa_4 = E\varepsilon_1^4 - 3 , \ \hat{\varphi}_0 = \int_0^\pi \varphi(\lambda) d\lambda \ \text{ and } \ \hat{\varphi}_r = 2\int_0^\pi \varphi(\lambda)\cos(r\lambda)d\lambda, r \geq 1.$$

This concludes the proof of Theorem 3.1, since

$$\int_0^\pi \varphi(\lambda) f(\lambda) d\lambda = \int_0^\pi \varphi(\lambda) \frac{1}{2\pi} \left\{\gamma(0) + 2\sum_{r=1}^{\infty} \gamma(r)\cos(\lambda r)\right\} d\lambda = \frac{1}{2\pi} \sum_{r=0}^{\infty} \hat{\varphi}_r \gamma(r)$$

and

$$\begin{aligned}& 2\pi \int_0^\pi \varphi^2(\lambda) f^2(\lambda) d\lambda \\ = \ & \frac{1}{\pi^2} \sum_{r,s=0}^{\infty} \hat{\varphi}_r \hat{\varphi}_s \int_0^\pi \cos(r\lambda)\cos(s\lambda)\{\gamma(0) + 2\sum_{k=1}^{\infty} \gamma(k)\cos(k\lambda)\} f(\lambda) d\lambda \\ = \ & \frac{1}{4\pi^2} \sum_{r,s=0}^{\infty} \sum_{k=-\infty}^{\infty} \hat{\varphi}_r \hat{\varphi}_s \{\gamma(k)\gamma(k-r+s) + \gamma(k)\gamma(k+r+s)\} , \\ = \ & \frac{1}{4\pi^2} \sum_{r,s=0}^{\infty} \sum_{k=-\infty}^{\infty} \hat{\varphi}_r \hat{\varphi}_s \{\gamma(k)\gamma(k-r+s) + \gamma(k+s)\gamma(k-r)\} ,\end{aligned}$$

where, for the last equality, we have used the following addition formula of trigonometric functions:

$$\cos a \cos b \cos c = \frac{1}{4}\left(\cos(a+b-c) + \cos(b+c-a) + \cos(c+a-b) + \cos(a+b+c)\right) .$$

■

Lemma 7.3 *Assume (A1) and (A2). Then the periodogram $I_n^+(\lambda_j)$ defined in Step III of the bootstrap algorithm satisfies*

$$I_n^+(\lambda_j) = \hat{\sigma}^2(P) \left|1 + \sum_{\nu=1}^{\infty} \hat{\alpha}_\nu(P) e^{-i\lambda_j \nu}\right|^2 I_{n,\varepsilon^*}(\lambda_j) + R_n^+(\lambda_j), \tag{7.17}$$

where $\max_{\lambda_j \in [0,\pi]} E^*(R_n^+(\lambda_j))^2 = O_P(n^{-1})$.

Proof: Let $\lambda = \lambda_j$. Following the derivation of Brockwell and Davis (1991), p. 346, we get for the discrete Fourier transform $J_n^+(\lambda)$ of $\{X_t^*\}$ that

$$J_n^+(\lambda) = \sigma(P)(1 + \sum_{\nu=1}^{\infty} \hat{\alpha}_\nu(P) e^{-i\lambda_j \nu}) J_{\varepsilon^*}(\lambda) + \sigma(P) Y_n^+(\lambda) \tag{7.18}$$

where

$$J_{\varepsilon^*}(\lambda) = n^{-1/2} \sum_{t=1}^{n} \varepsilon_t^* e^{-i\lambda t}, \tag{7.19}$$

$$Y_n^+(\lambda) = n^{-1/2} \sum_{\nu=0}^{\infty} \hat{\alpha}_\nu(P) e^{-\lambda\nu} U_{n,\nu}^+(\lambda) \tag{7.20}$$

and

$$U_{n,\nu}^+(\lambda) = \sum_{t=1-\nu}^{n-\nu} \varepsilon_t^* e^{-\lambda t} - \sum_{t=1}^{n} \varepsilon_t^* e^{-\lambda t}. \tag{7.21}$$

Since $I_n^+(\lambda) = (2\pi)^{-1} |J_n^+(\lambda)|^2$ we have obtained expression (7.17) and the remainder $R_n^+(\lambda)$ is given by

$$\begin{aligned} R_n^+(\lambda) &= \hat{\sigma}^2(P)(1 + \sum_{\nu=1}^{\infty} \hat{\alpha}_\nu(P) e^{-i\lambda_j \nu}) J_{\varepsilon^*}(\lambda) Y_n^+(-\lambda) \\ &\quad + \hat{\sigma}^2(P)(1 + \sum_{\nu=1}^{\infty} \hat{\alpha}_\nu(P) e^{i\lambda_j \nu}) J_{\varepsilon^*}(-\lambda) Y_n^+(\lambda) + \hat{\sigma}^2(P) |Y_n^+(\lambda)|^2. \end{aligned} \tag{7.22}$$

Note that for P fixed, $\hat{\mathbf{a}}(P)$ is a $\sqrt{n}$-consistent estimator of $\mathbf{a}(P)$; cf. Dahlhaus and Wefelmeyer (1996). Since $E^*(\varepsilon_1^*)^2 = 1$ we get then using Lemma 7.2 and the bound $E^*|U_{n,\nu}^+|^4 \leq 2|\nu| E^*(\varepsilon_1^*)^4 + 12|\nu|^2$, see Brockwell and Davis (1991), p. 347, that

$$\begin{aligned} E^*|Y_n^+(\lambda)|^4 &\leq n^{-2} \Big(\sum_{\nu=o}^{\infty} |\hat{\alpha}_\nu(P)| (2|\nu| E^*(\varepsilon_1^*)^4 + 12|\nu|^2)^{1/4} \Big)^4 \\ &= O_P(n^{-2}). \end{aligned} \tag{7.23}$$

Using expression (7.22) the assertion that $\max_{\lambda_j \in [0,\pi]} E^*(R_n^+(\lambda_j))^2 = O_P(n^{-1})$ follows then by the Cauchy-Schwarz inequality and taking into account Lemma 7.2, the fact that $E^*|J_{\varepsilon^*}(\lambda)|^2 = (2\pi)^{-1}$ and the bound (7.23). ∎

To prove Theorem 4.1 we use the decomposition

$$\begin{aligned} \sqrt{nb}\,(\hat{f}^*(\lambda) - \tilde{f}(\lambda)) &= \sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j) \hat{q}(\lambda_j) \Big(I_n^+(\lambda_j) - E^*(I_n^+(\lambda_j)) \Big) \\ &\quad + \sqrt{nb} \Big(\frac{1}{n} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j) \hat{q}(\lambda_j) E^*(I_n^+(\lambda_j)) - \hat{q}(\lambda) \hat{f}_{AR}(\lambda) \Big) \\ &= L_n^*(\lambda) + B_n^*(\lambda) \end{aligned} \tag{7.24}$$

where $K_b(\cdot) = b^{-1} K(\cdot/b)$ and an obvious notation for $L_n^*(\lambda)$ and $B_n^*(\lambda)$. The following two lemmas can then be established.

Lemma 7.4 *Assume (A1)-(A6) and let $P \in \mathbb{N}$ be fixed. If $n \to \infty$ then*

$$d_2\Big(\mathcal{L}(\sqrt{nb}\ (\hat{f}(\lambda) - E(\hat{f}(\lambda))),\ \mathcal{L}(L_n^*(\lambda)|X_1, X_2, \ldots, X_n)\Big) \to 0$$

in probability.

Proof: Since convergence in the d_2 metric is equivalent to weak convergence and convergence of the first two moments (cf. Bickel and Freedman (1981), Lemma 8.3) it suffices to show that

$$\mathrm{E}^*(L_n^*(\lambda))^2 \to \tau^2(\lambda) := f^2(\lambda)\frac{1}{2\pi}\int K^2(x)dx \tag{7.25}$$

and

$$\mathcal{L}(L_n^*(\lambda)|X_1, X_2, \ldots, X_n) \Rightarrow N(0, \tau^2(\lambda)) \tag{7.26}$$

in probability. Recall that $nb\mathrm{Var}(\hat{f}(\lambda)) \to \tau^2(\lambda)$ and $\sqrt{nb}\ (\hat{f}(\lambda) - E\hat{f}(\lambda)) \Rightarrow N(0, \tau^2(\lambda))$; cf. Anderson (1971), for a different but asymptotically equivalent estimator.

Consider first (7.25). We have

$$\begin{aligned} E^*(L_n^*(\lambda))^2 &= \frac{b}{n}\sum_{j=-N}^{N} K_b^2(\lambda - \lambda_j)\hat{q}^2(\lambda_j)\hat{f}_{AR}^2(\lambda_j)E^*(I_{n,\varepsilon^*}(\lambda_j) - 1)^2 + O_P(b) \\ &= \frac{b}{2\pi}\int K_b^2(\lambda - x)\hat{q}^2(x)\hat{f}_{AR}^2(x)dx + O_P(b) \\ &\to q^2(\lambda)f_{AR}^2(\lambda)\frac{1}{2\pi}\int K^2(x)dx \end{aligned}$$

in probability, by the continuity of the functions q and f_{AR} and the uniform convergences

$$\sup_{\lambda\in[0,\pi]} |\hat{f}_{AR}(\lambda) - f_{AR}(\lambda)| \to 0 \tag{7.27}$$

and

$$\sup_{\lambda\in[0,\pi]} |\hat{q}(\lambda) - q(\lambda)| \to 0 \tag{7.28}$$

in probability. To see (7.27) and (7.28) recall that for P fixed, $\hat{\mathbf{a}}(P)$ is a $\sqrt{n}$-consistent estimator of $\mathbf{a}(P)$. By a standard Taylor series argument and the continuity of the derivative it is easily seen that

$$\sup_{\lambda\in[0,\pi]} |\hat{f}_{AR}(\lambda) - f_{AR}(\lambda)| = O_P(n^{-1/2}).$$

By the above equality and because $\sup_{\lambda\in[0,\pi]} |(\hat{f}_{AR}(\lambda))^{-1}| = O_P(1)$ we get

$$\hat{q}(\lambda) = \frac{1}{nh}\sum_{j=-N}^{N} K\Big(\frac{\lambda - \lambda_j}{h}\Big)\frac{I_n(\lambda_j)}{f_{AR}(\lambda_j)} + O_P(n^{-1/2}), \tag{7.29}$$

where the $O_P(n^{-1/2})$ term is uniformly in $\lambda \in [0, \pi]$. The uniform consistency of the first term on the right hand side of (7.29) as an estimator of q follows then by standard arguments, cf. for instance the proof of Theorem A1 of Franke ad Härdle (1992).

We next show (7.26). For this note first that by Lemma 7.3

$$\sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j)\hat{q}(\lambda_j)R_n^+(\lambda_j = O_P(\sqrt{b}\ )$$

and, therefore,

$$L_n^*(\lambda) = \sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j)\tilde{f}(\lambda_j)(I_{n,\varepsilon^*}(\lambda_j) - 1) + o_P(1). \tag{7.30}$$

Now instead of the first term on the right hand sight of the above equality we consider in the following the asymptotically equivalent statistic

$$\sqrt{nb}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} K_b(\lambda - x)\tilde{f}(x)\Big(I_{\varepsilon^*}(x) - 1\Big)dx, \tag{7.31}$$

which appears by approximating the Riemann sum in (7.30) by the corresponding integral; cf. Brillinger (1981), Th. 5.9.1. We then have

$$\begin{aligned}
&\sqrt{nb}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} K_b(\lambda - x)\tilde{f}(x)\Big(I_{\varepsilon^*}(x) - 1\Big)dx \\
&\qquad = \sqrt{nb}\frac{1}{2\pi} \int_{-\pi}^{\pi} K(u)\tilde{f}(\lambda - ub)\Big(I_{\varepsilon^*}(\lambda - ub) - 1\Big)du \\
&\qquad = \tilde{f}(\lambda)\sqrt{nb}\frac{1}{2\pi} \int_{-\pi}^{\pi} K(u)\Big(I_{\varepsilon^*}(\lambda - ub) - 1\Big)du + D_n^+(\lambda)
\end{aligned}$$

where

$$D_n^+(\lambda) = \sqrt{nb}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} K(u)\Big(\tilde{f}(\lambda - ub) - \tilde{f}(\lambda)\Big)\Big(I_{\varepsilon^*}(\lambda - ub) - 1\Big)du.$$

Straightforward calculations yields that $E^*(D_n^+(\lambda)) = 0$ and $E^*(D_n^+(\lambda))^2 = O_P(|\tilde{f}(\lambda - ub) - \tilde{f}(\lambda)|^2) = O_P(b^2)$, where the last assertion follows by the uniform convergence of $\tilde{f}$ and (A3).

Thus in order to establish the asymptotic distribution of $L_n^*(\lambda)$ it suffices to consider the distribution of the asymptotically equivalent statistic

$$\tilde{f}(\lambda)\sqrt{nb}\ \frac{1}{2\pi} \int_{-\pi}^{\pi} K(u)\Big(I_{\varepsilon^*}(\lambda - ub) - 1\Big)du\ =\ \tilde{f}(\lambda)L_{1,n}^*(\lambda) \tag{7.32}$$

with an obvious notation for $L_{1,n}^*(\lambda)$. Substituting in $L_{1,n}^*(\lambda)$ the expression $I_{\varepsilon^*}(\lambda) = \sum_{s=-n+1}^{n-1} \hat{\gamma}_{\varepsilon^*}(s)\cos(s\lambda)$, where $\hat{\gamma}_{\varepsilon^*}(s) = n^{-1}\sum_{t=1}^{n-s} \varepsilon_t^*\varepsilon_{t+s}^*$, we get

$$\begin{aligned}
L_{1,n}^*(\lambda) &= 2\sqrt{nb}\sum_{s=1}^{n-1}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} K(u)\cos(s(\lambda - ub))du\Big)\hat{\gamma}_{\varepsilon^*}(s) \\
&\qquad + \sqrt{nb}(\hat{\gamma}_{\varepsilon^*}(0) - 1)\frac{1}{2\pi}\int_{-\pi}^{\pi} K(u)du \\
&= 2\sqrt{nb}\sum_{s=1}^{n-1}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} K(u)\cos(sub))du\Big)\cos(s\lambda)\hat{\gamma}_{\varepsilon^*}(s) + O_P(\sqrt{b}\ ) \\
&= 2\sqrt{nb}\sum_{s=1}^{n-1} k(sb)\cos(s\lambda)\hat{\gamma}_{\varepsilon^*}(s) + o_P(1)
\end{aligned}$$

where the last equality follows by (A4). Using the definition $m_n = [1/b]$ and substituting for $\hat{\gamma}_{\varepsilon^*}(s)$, the first term on the right hand side of the last equality above has the same distribution as

$$\begin{aligned}\sqrt{\frac{n}{m_n}}\sum_{s=1}^{m_n} k(s/m_n)\cos(s\lambda)\hat{\gamma}_{\varepsilon^*}(s) &= \frac{1}{\sqrt{nm_n}}\sum_{s=1}^{m_n} k(s/m_n)\cos(s\lambda)\sum_{t=1}^{n-s}\varepsilon_t^*\varepsilon_{t+s}^* \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n-1}\Big(\frac{1}{\sqrt{m_n}}\sum_{s=1}^{\min\{m_n,n-t\}} k(s/m_n)\cos(s\lambda)\varepsilon_t^*\varepsilon_{t+s}^*\Big).\end{aligned}$$

Let

$$W_{t,n}^+ = \frac{1}{\sqrt{m_n}}\sum_{s=1}^{m_n} k(s/m_n)\cos(s\lambda)\varepsilon_t^*\varepsilon_{t+s}^*$$

and verify that

$$\Big|\frac{1}{\sqrt{n}}\sum_{t=1}^{n-1}\Big(\frac{1}{\sqrt{m_n}}\sum_{s=1}^{\min\{m_n,n-t\}} k(s/m_n)\cos(s\lambda)\varepsilon_t^*\varepsilon_{t+s}^*\Big) - \frac{1}{\sqrt{n}}\sum_{t=1}^{n-1} W_{t,n}^+\Big| = O_P\Big(\sqrt{\frac{m_n}{n}}\Big).$$

Since $m_n/n \to 0$ we can restrict or considerations to the statistic $n^{-1/2}\sum_{t=1}^{n-1} W_{t,n}^+$. To obtain the asymptotic distribution of this statistic, let $l_n = [\sqrt{nm_n}]$, $M_n = [n/l_n]$ and consider for $j = 1, 2, \ldots, M_n$ the random variables

$$Z_{j,n}^+ = \frac{1}{\sqrt{l_n}}\Big(W_{(j-1)l_n+1,n}^+ + W_{(j-1)l_n+2,n}^+ + \ldots + W_{jl_n-m_n,n}^+\Big).$$

Note that $E^*(Z_{j,n}^+) = 0$ and $E^*(Z_{j,n}^+)^2 = \big(1 - m_n/l_n\big)E(W_{j,n}^+)^2$. Furthermore, by construction the $Z_{j,n}^+$'s are independent since they are sums of nonoverlapping segments of the i.i.d. series $\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_n^*$. It is easily seen that $\sum_{t=1}^{n-1} W_{t,n}^+ - \sqrt{l_n}\sum_{j=1}^{M_n} Z_{j,n}^+ = O_P(M_n m_n)$ which implies that

$$E^*\Big(\frac{1}{\sqrt{n}}\sum_{t=1}^{n-1} W_{t,n}^+ - \frac{1}{\sqrt{M_n}}\sum_{j=1}^{M_n} Z_{j,n}^+\Big)^2 = O_P(m_m l_n^{-1}) \to 0$$

in probability by (A6) and the definition of l_n. Now, since $\{Z_{j,n}^+, 1 \le j \le M_n; n = 1, 2, \ldots\}$ forms a triangular array of independent random variables and $Var^*(M_n^{-1/2}\sum_{j=1}^{M_n} Z_{j,n}^+) = O_P(1)$, the desired asymptotic normality follows from Lyaponov's condition (cf. Shiryayev (1984)) if we show that $M_n^{-2}\sum_{j=1}^{M_n} E^*(Z_{j,n}^+)^4 \to 0$. Note that

$$E^*(Z_{j,n}^+)^4 = l_n^{-2}\sum_{t=1}^{l_n-m_n}\sum_{s=1}^{l_n-m_n}\sum_{r=1}^{l_n-m_n}\sum_{q=1}^{l_n-m_n} E^*(W_{t,n}^+ W_{s,n}^+ W_{r,n}^+ W_{q,n}^+) \tag{7.33}$$

and using the definition of $W_{t,n}^+$, evaluation of (7.33) requires evaluation of the expectation $E^*(\varepsilon_t^*\varepsilon_{t+k_1}^*\varepsilon_s^*\varepsilon_{s+k_1}^*\varepsilon_r^*\varepsilon_{r+k_1}^*\varepsilon_q^*\varepsilon_{q+k_1}^*)$. Evaluating this expectation using the independence of ε_t^+ we get after some tedious calculations the following bounds for the different nonvanishing terms in (7.33):

For the case where all indices are equal, i.e., the case $t = s = r = q$, we have

$$\begin{aligned}\frac{1}{l_n^2}\sum_{t=1}^{l_n-m_n} E^*(W_{t,n}^+)^4 &= \frac{l_n-m_n}{l_n^2 m_n^2}\Big\{E^*(\varepsilon_1^*)^4\sum_{s=1}^{m_n} k^4(s/m_n)\cos^4(s\lambda) \\ &\quad +3(E^*(\varepsilon_1^*)^2)^2 E^*(\varepsilon_1^*)^4\Big(\sum_{s=1}^{m_n} k^2(s/m_n)\cos(s\lambda)\Big)^2\Big\} \\ &\le O_P((l_n-m_n)l_n^{-2}m_n^{-1}) + O_P((l_n-m_n)l_n^{-2}). \qquad (7.34)\end{aligned}$$

There are four terms of the form

$$\begin{aligned}&\frac{1}{l_n^2}\sum_{t_1=1}^{l_n-m_n}\sum_{t_2=1,t_1\neq t_2}^{l_n-m_n} E^*\Big((W_{t_1,n}^+)^3 W_{t_2,n}^+\Big) \\ &= \frac{1}{l_n^2 m_n^2}\sum_{t_1=1}^{l_n-m_n}\sum_{t_2=1,t_1\neq t_2}^{l_n-m_n} E^*\Big(\sum_{s_1}^{m_n} k(s_1/m_n)\cos(s_1\lambda)\varepsilon_{t_1}^*\varepsilon_{t_1+s_1}^*\Big)^3 \\ &\quad \times\Big(\sum_{s_2}^{m_n} k(s_2/m_n)\cos(s_2\lambda)\varepsilon_{t_2}^*\varepsilon_{t_2+s_2}^*\Big) \\ &\le (E^*(\varepsilon_1^*)^3)^2 E^*(\varepsilon_1^*)^2\, O\Big(\frac{(l_n-m_n)(l_n-m_n-1)}{m_n l_n^2}\Big) \\ &= O_P\Big(\frac{(l_n-m_n)(l_n-m_n-1)}{m_n l_n^2}\Big). \qquad (7.35)\end{aligned}$$

Furthermore, there are three terms of the form

$$\begin{aligned}&\frac{1}{l_n^2}\sum_{t_1=1}^{l_n-m_n}\sum_{t_2=1,t_1\neq t_2}^{l_n-m_n} E^*\Big((W_{t_1,n}^+)^2 (W_{t_2,n}^+)^2\Big) \\ &= \frac{1}{l_n^2 m_n^2}\sum_{t_1=1}^{l_n-m_n}\sum_{t_2=1,t_1\neq t_2}^{l_n-m_n} E^*\Big(\sum_{s_1}^{m_n} k^2(s_1/m_n)\cos^2(s_1\lambda)\varepsilon_{t_1}^{*2}\varepsilon_{t_1+s_1}^{*2}\Big) \\ &\quad \times\Big(\sum_{s_2}^{m_n} k^2(s_2/m_n)\cos^2(s_2\lambda)\varepsilon_{t_2}^{*2}\varepsilon_{t_2+s_2}^{*2}\Big) \\ &= \Big(E^*(\varepsilon_1^*)^2\Big)^4\, O\Big(\frac{(l_n-m_n)(l_n-m_n-1)}{l_n^2}\Big) \\ &= O_P(1). \qquad (7.36)\end{aligned}$$

Finally, there are six terms of the form

$$\begin{aligned}&\frac{1}{l_n^2}\sum_{t_1=1}^{l_n-m_n}\sum_{t_2=1}^{l_n-m_n}\sum_{t_3=1,t_1\neq t_2,t_2\neq t_3,t_1\neq t_3}^{l_n-m_n} E^*\Big((W_{t_1,n}^+)^2 W_{t_2,n}^+ W_{t_3,n}^+\Big) \qquad (7.37) \\ &= O\Big(\frac{(l_n-m_n)(l_n-m_n-1)m_n}{l_n^2}\Big)O_P(m_n^{-1}) \\ &= O_P(1), \qquad (7.38)\end{aligned}$$

where the equality before the last one follows because there are $(l_n-m_n)(l_n-m_n-1)m_n$ nonvanishing cases in evaluating (7.37) and

$$\Big|E^*((W_{t_1,n}^+)^2 W_{t_2,n}^+ W_{t_3,n}^+)\Big| \le m_n^{-1}\Big(E^*(\varepsilon_1^*)^2\Big)^4 O(1).$$

Now, (7.34), (7.35), (7.36)) and (7.38) imply that

$$\frac{1}{M_n^2}\sum_{j=1}^{M_n} E^*(Z_{j,n}^+)^4 = O_P(M_n^{-1}) \to 0,$$

which concludes the proof of the lemma. ∎

Lemma 7.5 *Assume (A1)-(A6), (A8)-(A9) and let $P \in \mathbb{N}$ be fixed. If $b \sim n^{1/5}$ and $nh^3 \to \infty$ as $n \to \infty$, then*

$$B_n^*(\lambda) \to \frac{1}{4\pi} f''(\lambda) \int u^2 K(u) du$$

in probability.

Proof: Using Lemma 7.3 and the fact that $|n^{-1}\sum_{j=1}^n K_b(\lambda - \lambda_j) - 1| = O(n^{-1}b^{-1})$ we get

$$\begin{aligned}
B_n^*(\lambda) &= \sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j)\Big(\hat{q}(\lambda_j \hat{f}_{AR}(\lambda_j) - \hat{q}(\lambda)\hat{f}_{AR}(\lambda)\Big) \\
&\qquad + O_P(n^{-1/2}b^{-1/2}) + O_P(\sqrt{b}\,) \\
&= \hat{f}_{AR}(\lambda)\sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j)\Big(\hat{q}(\lambda_j) - \hat{q}(\lambda)\Big) \\
&\quad + \sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j)\Big(\hat{q}(\lambda_j) - \hat{q}(\lambda)\Big)\Big(\hat{f}_{AR}(\lambda_j) - \hat{f}_{AR}(\lambda)\Big) \\
&\quad + \hat{q}(\lambda)\sqrt{\frac{b}{n}} \sum_{j=-N}^{N} K_b(\lambda - \lambda_j)\Big(\hat{f}_{AR}(\lambda_j) - \hat{f}_{AR}(\lambda)\Big) + O_P(n^{-1/2}b^{-1/2}) + O_P(\sqrt{b}\,) \\
&= B_{1,n}^*(\lambda) + B_{2,n}^*(\lambda) + B_{3,n}^*(\lambda) + O_P(n^{-1/2}b^{-1/2}) + O_P(\sqrt{b}\,) \qquad (7.39)
\end{aligned}$$

with an obvious notation for $B_{i,n}^*(\lambda)$, $i = 1, 2, 3$. To establish the desired convergence it suffices to show that

$$B_{1,n}^*(\lambda) \to \frac{1}{4\pi} f_{AR}(\lambda) q''(\lambda) \int_{-\pi}^{\pi} u^2 K(u) du, \qquad (7.40)$$

$$B_{2,n}^*(\lambda) \to \frac{1}{2\pi} f'_{AR}(\lambda) q'(\lambda) \int_{-\pi}^{\pi} u^2 K(u) du \qquad (7.41)$$

and

$$B_{3,n}^*(\lambda) \to \frac{1}{4\pi} f''_{AR}(\lambda) q(\lambda) \int_{-\pi}^{\pi} u^2 K(u) du, \qquad (7.42)$$

in probability since $f''(\lambda) = f_{AR}(\lambda)q''(\lambda) + 2f'_{AR}(\lambda)q'(\lambda) + f''_{AR}(\lambda)q(\lambda)$. We proceed by showing that (7.40) to (7.42) are true.

To prove (7.40) note that by (A8) and a Taylor series expansion we get

$$\begin{aligned}
&\sqrt{\frac{b}{n}}\sum_{j=-N}^{N} K_b(\lambda-\lambda_j)\big(\hat{q}(\lambda_j)-\hat{q}(\lambda)\big)\\
&= \frac{1}{n^{3/2}b^{1/2}h}\sum_{j=-N}^{N}\sum_{s=-N}^{N} K\big(\frac{\lambda-\lambda_j}{b}\big)\Big[K\big(\frac{\lambda_j-\lambda_s}{h}\big)-K\big(\frac{\lambda-\lambda_s}{h}\big)\Big]\frac{I_n(\lambda_s)}{\hat{f}_{AR}(\lambda_s)}\\
&= \frac{1}{n^{3/2}b^{1/2}h}\sum_{j=-N}^{N}\sum_{s=-N}^{N} K\big(\frac{\lambda-\lambda_j}{b}\big)\frac{\lambda_j-\lambda}{h}K'\big(\frac{\lambda-\lambda_s}{h}\big)\frac{I_n(\lambda_s)}{\hat{f}_{AR}(\lambda_s)}\\
&\quad+\frac{1}{2n^{3/2}b^{1/2}h}\sum_{j=-N}^{N}\sum_{s=-N}^{N} K\big(\frac{\lambda-\lambda_j}{b}\big)\big(\frac{\lambda_j-\lambda}{h}\big)^2K''\big(\frac{\lambda-\lambda_s}{h}\big)\frac{I_n(\lambda_s)}{\hat{f}_{AR}(\lambda_s)}\\
&\quad+O_P\big(n^{1/2}b^{7/2}\big). \qquad (7.43)
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{1}{n^{3/2}b^{1/2}h}\sum_{j=-N}^{N}\sum_{s=-N}^{N} K\big(\frac{\lambda-\lambda_j}{b}\big)\frac{\lambda_j-\lambda}{h}K'\big(\frac{\lambda-\lambda_s}{h}\big)\frac{I_n(\lambda_s)}{f_{AR}(\lambda_s)}\\
&= \frac{\sqrt{n}}{2\pi\sqrt{b}}\int_{-\pi}^{\pi}(x-\lambda)K\big(\frac{\lambda-x}{b}\big)dx\frac{1}{nh^2}\sum_{s=-N}^{N}K'\big(\frac{\lambda-\lambda_s}{h}\big)\frac{I_n(\lambda_s)}{f_{AR}(\lambda_s)}+o_P(1)\\
&= o_P(1), \qquad (7.44)
\end{aligned}$$

using $\int uK(u)du=0$ and the fact that the second multiplicative term on the right hand side of the equality before the last one is $O_P(1)$ because it converges to $q^{'}(\lambda)$. Similarly,

$$\begin{aligned}
&\frac{1}{2n^{3/2}b^{1/2}h}\sum_{j=-N}^{N}\sum_{s=-N}^{N} K\big(\frac{\lambda-\lambda_j}{b}\big)\big(\frac{\lambda_j-\lambda}{h}\big)^2K''\big(\frac{\lambda-\lambda_s}{h}\big)\frac{I_n(\lambda_s)}{f_{AR}(\lambda_s)}\\
&= \frac{1}{2}\frac{1}{\sqrt{nb}}\sum_{j=-N}^{N}(\lambda_j-\lambda)^2K\big(\frac{\lambda-\lambda_j}{b}\big)\frac{1}{nh^3}\sum_{s=-N}^{N}K''\big(\frac{\lambda-\lambda_s}{h}\big)\frac{I_n(\lambda_s)}{f_{AR}(\lambda_s)}\\
&\to \frac{1}{4\pi}\int_{\pi}^{\pi}u^2K(u)du\;q^{''}(\lambda), \qquad (7.45)
\end{aligned}$$

in probability. Note that the last assertion follows using $b\sim n^{-1/5}$ and because for $nh^3\to\infty$ we have $n^{-1}h^{-3}\sum_{s=-N}^{N}K''((\lambda-\lambda_s)/h)I_n(\lambda_s)/f_{AR}(\lambda_s)\to q^{''}(\lambda)$ in probability.

Now, by (7.43), (7.44) and (7.45) and because $\hat{f}_{AR}(\lambda)=f_{AR}(\lambda)+O_P(n^{-1/2})$ uniformly in λ, we obtain (7.40).

Since (7.41) and (7.42) follows using similar arguments we stress only the essentials.

For $B^*_{2,n}(\lambda)$ we have using the differentiability of $\hat{f}_{AR}(\lambda)$ with respect to λ and similar arguments as in obtaining (7.43) that

$$B^*_{2,n}(\lambda) = \hat{f}^{'}_{AR}(\lambda)\frac{1}{\sqrt{nb}}\sum_{j=-N}^{N}K\big(\frac{\lambda-\lambda_j}{b}\big)(\lambda_j-\lambda)(\hat{q}(\lambda_j)-q(\lambda))+o_P(1)$$

$$
\begin{aligned}
&= \hat{f}'_{AR}(\lambda)\frac{1}{\sqrt{nb}}\sum_{j=-N}^{N} K\Big(\frac{\lambda-\lambda_j}{b}\Big)(\lambda_j-\lambda)^2\frac{1}{nh^2}\sum_{s=-N}^{N} K'\Big(\frac{\lambda-\lambda_s}{h}\Big)\frac{I_n(\lambda_s)}{\hat{f}_{AR}(\lambda_s)} + o_P(1)\\
&\to f'_{AR}(\lambda)q'(\lambda)\frac{1}{2\pi}\int_{-\pi}^{\pi} u^2K(u)du,
\end{aligned}
$$

in probability. To show (7.42) we use a Taylor series expansion of $\hat{f}_{AR}(\lambda_j)$ around $\hat{f}_{AR}(\lambda)$ and obtain

$$
\begin{aligned}
B^*_{3,n}(\lambda) &= \hat{q}(\lambda)\hat{f}'_{AR}(\lambda)\frac{1}{\sqrt{nb}}\sum_{j=-N}^{N} K\Big(\frac{\lambda-\lambda_j}{b}\Big)(\lambda_j-\lambda)\\
&\quad +\frac{1}{2}\hat{q}(\lambda)\hat{f}''_{AR}(\lambda)\frac{1}{\sqrt{nb}}\sum_{j=-N}^{N} K\Big(\frac{\lambda-\lambda_j}{b}\Big)(\lambda_j-\lambda)^2 + o_P(1)\\
&\to q(\lambda)f''_{AR}(\lambda)\frac{1}{4\pi}\int_{-\pi}^{\pi} u^2K(u)du.
\end{aligned}
$$

■

Proof of Theorem 4.1: Consider part (ii) of the theorem. By Lemma 8.8 of Bickel and Freedman (1981), we can split the squared Mallows metric into a variance part $V_n^2(\lambda)$ and a squared bias part $b_n^2(\lambda)$, where

$$
V_n^2(\lambda) = d_2^2\Big(\mathcal{L}(\sqrt{nb}(\hat{f}(\lambda)-E\hat{f}(\lambda)), \mathcal{L}(\sqrt{nb}(\hat{f}^*(\lambda)-E^*\hat{f}^*(\lambda))|X_1,X_2,\ldots,X_n)\Big)
$$

and

$$
b_n^2(\lambda) = nb\Big|(E\hat{f}(\lambda))-f(\lambda)) - (E^*\hat{f}^*(\lambda))-\widetilde{f}(\lambda))\Big|^2.
$$

By Lemma 7.4 and 7.5 we then have that $V_n^2(\lambda)\to 0$ and $b_n^2(\lambda)\to 0$ in probability. Part (i) of the theorem follows by the same arguments but by ignoring the bias term. ■

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		Mean($\hat{\sigma}_1^*$)	SD($\hat{\sigma}_1^*$)	MSE($\hat{\sigma}_1^*$)$\times 10^3$
Est. Exact		0.103		
AR-Boot	P=1	0.132	0.00748	0.8969
	P=3	0.099	0.01466	0.2309
	P=5	0.105	0.01525	0.2367
	P_{AIC}	0.107	0.01640	0.2849
NP-Boot	h=0.1	0.078	0.01635	0.8923
	h=0.2	0.093	0.01897	0.4598
	h=0.3	0.103	0.01807	0.3265
	h_{CV}	0.099	0.01764	0.3314
ARAP-Boot	P=1	0.104	0.01331	0.1781
	P=3	0.101	0.01504	0.2302
	P=5	0.105	0.01657	0.2785
	P_{AIC}	0.106	0.01616	0.2701

Table 1:
Autoregressive Bootstrap (AR-Boot), Nonparametric Periodogram Bootstrap (NP-Boot) and Autoregressive-Aided Periodogram Bootstrap (ARAP-Boot) estimates of the standard deviation of the first order sample autocorrelation.