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Walrasian Equilibria in a Production Economy with Indivisibilities

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Abstract

This paper studies a production economy with indivisibilities. We provide a characterization for all Walrasian equilibria and a necessary and sufficient condition for the existence of Walrasian equilibrium. We find a sufficient condition for every descending (ascending) price process to converge to a Walrasian equilibrium within finite periods of time.

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1 Introduction

Firms cut or raise their prices in response to not only the market demand and supply of their own products but also the demand and supply of the products of the other firms. But firms rarely cut or raise their prices once for all. Instead they often cut or raise their prices periodically. This type of pricing behavior invites an interesting question: Does the price path in an economy eventually end up with an equilibrium? This question is not so obvious when firms' products are interdependent with each other. On the other hand, an answer to the question has a fundamental implication for equilibrium in Macroeconomics which requires that aggregate demand (AD) equal aggregate supply (AS). The equilibrium condition in Macroeconomics implies that the demand and supply for each product in an economy is balanced at the given equilibrium price level at the same time. Macroeconomists often use the AD-AS analysis without doubts whether an equilibrium exists and how such an equilibrium is achieved. If such an equilibrium does not exist, there always exists some products that are overdemanded or supplied no matter what the price level is in an economy. This existence issue may not be problematic when each product is assumed perfectly divisible and all products are independent with each other. But the existence issue will pop up when a product is indivisible and products in an economy are interdependent with each other in one way or another. Moreover, even though an equilibrium exists for a production economy, not every price path converges to an equilibrium. Some price paths may stalk with complex cycles. Therefore, how should firms cut or raise their prices such that their prices eventually end up with an equilibrium price level that balances the aggregate demand and supply in an economy?

This paper considers a production economy with indivisibilities to provide some insights to these questions. The production economy consists of a finite number of consumers and firms. There are a finite number of indivisible (differentiated) products in the economy. A commodity bundle in the economy may consist of different numbers of different products with multiple units of each product. A consumer or a firm may consume or produce any commodity bundle as he or she wishes. Therefore, a consumer is characterized by his (interdependent) utility function over all possible commodity bundles in the economy and a firm is characterized by its (interdependent) cost function over all possible commodity bundles. A feasible allocation in the economy is a collection of commodity bundles, one commodity bundle for each consumer or producer, such that aggregate demand does not exceed aggregate supply. A feasible allocation leaves the possibility that some units of a product may not be consumed; but what is consumed should be produced in the economy. A Walrasian equilibrium in this economy consists of a price vector (over the products) and a feasible

allocation such that each commodity bundle for each consumer gives him the greatest consumer surplus and each commodity bundle for each firm gives her the greatest profit, and the market clearing condition is satisfied in the sense that aggregate demand is precisely equal to aggregate supply. Since all products are indivisible, the market clearing condition for the aggregate demand and supply also implies that the demand and supply for each individual product is balanced.

Not all such production economies have an equilibrium. Indeed Section 4 uses an example, adopted from Kelso and Crawford (1982), to show that a Walrasian equilibrium may not exist for some production economies. Therefore we first provide a full characterization of all Walrasian equilibria in Section 3 and derive a useful necessary and sufficient condition for the existence of Walrasian equilibrium in Section 4. Our major concern is the condition such that certain price processes converge to a Walrasian equilibrium. Precisely, we attempt to look for the conditions for every descending or ascending price path to converge to a Walrasian equilibrium within finite periods of time.

We identify one sufficient condition for every descending price path to converge to an equilibrium. We say that a product is overdemanded (oversupplied) with respect to a collection of commodity bundles if the number of that product supplied is not greater (smaller) than the number of the product demanded in that collection. A collection of commodity bundles has the aggregate overdemanded (oversupplied) property if each product is overdemanded (oversupplied) with respect to that collection. The condition for the descending price path says that each price vector at each period of time induces at least one collection of commodity bundles with the aggregate oversupplied property[1]; see Section 5 for the formal definition. Intuitively, firms, starting with high prices of their products that may well create oversupplies, should not cut their prices to the degree that creates (strictly) overdemands for every product under any collection of commodity bundles induced by the price vector. As long as firms practice this way, our result shows that the price paths in the economy eventually arrive at an equilibrium within finite periods of time.

The sufficient condition for the ascending price path is the opposite to that for the descending price path, as one may have expected. The condition says that the price vector at each period of time should induce at least one collection of commodity bundles with the aggregate overdemanded property. Under this condition, every ascending price path converges to a Walrasian equilibrium within finite periods of time. When prices are ascending, the condition excludes those price paths that may well create oversupplies for every product under any circumstance at some periods. Many dynamic auctions for the sale of single object are designed in such a way that in each intermediate

[1] Note that we are considering the descending price paths.

period of the auction some overdemanded exists. Moreover the overdemanded situations are eliminated by increasing the price or prices gradually. Our condition captures some merits behind this practice and it may provide some useful guidance in designing an auction for the sale of multiple products with multiple units of each product, when consumers' utility functions and producers' cost functions are both interdependent.

Our production economy is closely related to Kelso and Crawford (1982), Bikhchandani and Mamer (1994), Gul and Stacchetti (1996a,b) and Ma (1997a,b). Kelso and Crawford (1982) generalized the job-matching market in Crawford and Knoer (1981) to allow a firm to hire as many workers as he wishes and to have more complex utility functions. As in Crawford and Knoer (1981), they generalized the Gale and Shapley (1962) deferred proposal algorithm to their job-matching market. Under their gross substitutes condition, they showed that their generalized algorithm must stop within finite periods of time with a core outcome (thus a Walrasian equilibrium). Bikhchandani and Mamer (1994) considered an exchange economy without production and provided a necessary and sufficient condition for the existence of Walrasian equilibrium. Our characterization theorem in Section 3 and the necessary and sufficient condition for the existence of Walrasian equilibrium for the production economy are related to their condition in spirit. Gül and Stacchetti (1996a,b) studied the exchange economy as in Bikhchandani and Mamer (1994) and designed an English auction by a generalization of the Hall Theorem (Hall (1935), Gale (1962)). They provided two new sufficient conditions, the no complementarities and the single improvement property, for the existence of Walrasian equilibrium. They showed that the set of Walrasian prices forms a lattice under the common interests of sellers and the gross substitutes condition is equivalent to their two new conditions. Moreover their auction converges to the minimum Walrasian price vector within finite periods of time. Note that not all exchange economies have the minimum Walrasian equilibrium. Ma (1997a) studied the existence of Walrasian equilibrium in an exchange economy with personalized initial endowments by means of a coalitional form game and showed that the balancedness of the game may be a useful condition for the existence of Walrasian equilibrium. Ma (1997b) studied the conditions under which the English auction converges to a Walrasian equilibrium within finite periods of time. Meanwhile, Gul and Stacchetti (1996a,b) also considered the production economy and designed a double auction that eventually converges to the minimum Walrasian equilibrium when the utility and cost functions satisfy their no complementarity condition (or the gross substitutes condition in Kelso and Crawford (1982)).

These papers studied economies with fixed numbers of objects or workers. The total number of objects consumed or produced is exogenously given. We fix the number of products in the

production economy in this paper. But how many units of each product that will be produced or consumed in the economy are endogenously determined. The underlying commodity bundles consumed and produced along each equilibrium price path are path-dependent, though the set of products is the same along the path. Kelso and Crawford (1982) and Gul and Stacchetti (1996b) are interested in finding one equilibrium price path by designing a dynamic auction procedure. This paper is searching for the conditions such that every ascending or descending price process converges to a Walrasian equilibrium. Typically there are infinite numbers of price paths (if any) that satisfy our conditions in this paper.

The rest of the paper is organized as follows. Section 2 introduces the model and some definitions. Section 3 provides some preliminary results. Section 4 provides a necessary and sufficient condition for the existence of Walrasian equilibrium. Section 5 proves the convergence theorems. Section 6 presents an oligopoly economy with constant marginal costs in which there exists a unique Walrasian price vector. Section 7 concludes the paper with some unanswered issues.

2 The Model

Let $N = \{1, 2, \cdots, i, \cdots, n\}$ denote the set of consumers and $F = \{1, 2, \cdots, j, \cdots, m\}$ the set of firms in the economy. The set of all possible differentiated products in the economy is described by $\Omega = \{\omega_1, \cdots, \omega_l\}$. Consumers or producers may consume or produce any combinations of products with any number of each product. To capture this idea, we need some definitions. For any set X, define an unordered family of elements of X to be a collection of elements, not necessarily distinct, in which the order is immaterial (Roth and Sotomayor (1990)). Denote the set of all unordered families of elements of Ω by 3^{Ω}, which distinguishes it from the set of all subsets of Ω, 2^{Ω}, by the fact that an unordered family of elements of Ω may contain several copies of the same element. For example, $\{\omega_1, \omega_1, \omega_2\}$ is an unordered family of elements of Ω that contains two copies of the same product ω_1. Note that an unordered family of elements of a set X may contain an infinite number of the same element. An unordered family of elements of Ω is called a commodity bundle. Given a commodity bundle $A \in 3^{\Omega}$, each element in A is called an object. Given two commodity bundles $A, B \in 3^{\Omega}$, the union of $A \cup B$ consists of all objects in A and B. $A \subset B$ is understood as usual as that every object in A can be found in B. Whenever we write $a \in A$, it means any copy of a that is in A. $\#A$ denotes the number of objects in A.

For $i \in N$, consumer i is characterized by his utility function $u_i : 3^{\Omega} \to R$, satisfying $u_i(\emptyset) = 0$.

For $j \in F$, firm j is characterized by her cost function $c_j : 3^\Omega \to R$. $c_j(\{\emptyset\}) \geq 0$ is identified as the fixed cost of firm j. We assume that c_j is strictly monotone in the sense that for any two commodity bundles A and B such that $B \subset A$, $c_j(A) \geq c_j(B)$ and $c_j(A) > c_j(B)$ whenever $B \subset A$ and $A \neq B$. Consumers' utility functions are bounded above in the sense that $u(A) \leq M$ for all $A \in 3^\Omega$, where M is a finite number that may be quite large. A production economy with indivisibilities is described by

$$\mathcal{E} = ((N, u); (F, c); (\Omega, 3^\Omega)).$$

We borrow some definitions from Gul and Stacchetti (1996a). Let $p \in R^l_+$ be a price vector and $A \in 3^\Omega$ be a commodity bundle, define

$$< p, A >= \sum_{a \in A} p_a.$$

Definition: Consumer i's consumer surplus function $v_i : 3^\Omega \times R^l_+ \to R$ and demand correspondence $D_i : R^l_+ \to 3^\Omega$ are defined by

$$v_i(A, p) = u_i(A) - < p, A >, \ \ A \in 3^\Omega, \ \ p \in R^l_+$$

$$D_i(p) = \{A \in 3^\Omega \mid v_i(A, p) \geq v_i(B, p) \text{ for all } B \in 3^\Omega\}, \ \ p \in R^l_+.$$

Definition: Firm j's profit function $\pi_j : 3^\Omega \times R^l_+ \to R$ and supply correspondence $S_j : R^l_+ \to 3^\Omega$ are defined by

$$\pi_j(A, p) = < p, A > - c_j(A), \ \ A \in 3^\Omega, \ \ p \in R^l_+$$

$$S_j(p) = \{A \in 3^\Omega \mid \pi_j(A, p) \geq \pi_j(B, p) \text{ for all } B \in 3^\Omega\}, \ \ p \in R^l_+.$$

Definition: A *feasible allocation* $(X; Y) = (X_1, \cdots, X_i, \cdots, X_n; Y_1, \cdots, Y_j, \cdots, Y_m)$ is such that (a) $X_i, Y_j \in 3^\Omega$ for all $i \in N$ and $j \in F$ and (b) $\cup_{i \in N} X_i \subset \cup_{j \in F} Y_j$.

Given a feasible allocation $(X; Y)$, we define $\cup_{i \in N} X_i$ and $\cup_{j \in F} Y_j$ to be the *aggregate demand* and *supply* respectively in the economy.

Given a feasible allocation, a consumer i may consume a commodity bundle that contains several copies of the same product and a firm may produce a commodity bundle that contains many

copies of the same product. In fact X_i or Y_j may well contain an infinite number of the same product. Nevertheless what is consumed should not be beyond what is produced in the economy. This does not mean that the model excludes trade because foreign countries can be seen as firms in F.

Definition: A feasible allocation $(X;Y)$ is *optimal* if it is a solution of the following problem

$$V = \max\{\sum_{i\in N} u_i(X_i) - \sum_{j\in F} c_j(Y_j) \mid (X;Y) \text{ is a feasible allocation}\}.$$

V is the total social surplus of an economy. Note that given the assumptions on consumers' utility functions and firms' cost functions, V is a finite number. But two optimal allocations may be quite different from each other.

Definition: $(p;X;Y)$ is a *Walrasian equilibrium* if (a) (optimality) $X_i \in D_i(p)$ for all $i \in N$ and $Y_j \in S_j(p)$ for all $j \in F$ and (b) (market clearing condition) $\cup_{i\in N} X_i = \cup_{j\in F} Y_j$, aggregate demand equals aggregate supply.

Remark: In the definition of a Walrasian equilibrium in Gul and Stacchetti (1996b) with production, aggregate supply may exceed aggregate demand. Our definition excludes this possibility.

Example 1: Consider a simple economy with $\Omega = (a,b)$, $N = (1,2)$ and $F = (\alpha,\beta)$. The utility functions are as follows:

$$u_1(\{a\}) = 2, u_1(\{a,a\}) = 8, u_1(\{a,a,a\}) = 9,$$
$$u_1(\{b\}) = 3, u_1(\{b,b\}) = 6,$$
$$u_1(\{a,b\}) = 9, u_1(\{A\}) = 10 \text{ for any other } A \in 3^\Omega$$

$$u_2(\{a\}) = 1, u_2(\{a,a\}) = 3, u_2(\{a,a,a\}) = 5, u_2(\{a,a,a,a\}) = 7,$$
$$u_2(\{b\}) = 3, u_2(\{b,b\}) = 7, u_2(\{b,b,b\}) = 9,$$
$$u_2(\{a,b\}) = 4, u_2(\{a,a,b\}) = 9, u_2(A) = 10 \text{ for any other } A \in 3^\Omega$$

The cost functions are "additive" [2]:

$$c_\alpha(\emptyset) = 0, c_\alpha(A) = 2\#A \text{ for all } A \in 3^{\{a\}}$$

$$c_\alpha(A) = \infty \text{ for all } A \text{ such that } b \in A.$$

$$c_\beta(\emptyset) = 0, c_\beta(B) = 3\#B \text{ for all } B \in 3^{\{b\}}$$

$$c_\beta(B) = \infty \text{ for all } B \text{ such that } a \in B.$$

$$(X;Y) = (\{a,a\},\{a,a,b\};\{a,a,a,a\},\{b\})$$

and

$$(\bar{X};\bar{Y}) = (\{a,b\},\{a,a,b\};\{a,a,a\},\{b,b\})$$

are the two optimal allocations in this economy. The unique Walrasian price vector is $p = (2,3)$. p_a and p_b are the marginal costs of firms α and β respectively. $(p;X;Y)$ and $(p;\bar{X};\bar{Y})$ are the two Walrasian equilibria in this economy. For a general result, see Section 6. □

3 Some Preliminary Results

This section provides a characterization of all Walrasian equilibria by a hyperplane. Given any price vector $p \in R^l_+$, Lemma 1 shows that the sum of total consumer surpluses and total profits is no less than the total social surplus V. Denote $v_i(p) = v_i(A,p)$ for $A \in D_i(p)$ and $\pi_j(p) = \pi_j(A,p)$ for $A \in S_j(p)$, $p \in R^l_+$.

Lemma 1: For any $p \in R^l_+$,

$$\sum_{i \in N} v_i(p) + \sum_{j \in F} \pi_j(p) \geq V.$$

Proof: For all feasible allocations $(X;Y)$, we have that

$$v_i(p) \geq v_i(X_i,p), \text{ for all } i \in N$$

and

$$\pi_j(p) \geq v_i(Y_j,p), \text{ for all } j \in F.$$

[2]This makes the computation of the optimal allocations much easier.

Therefore,

$$\begin{aligned}\sum_{i\in N} v_i(p) + \sum_{j\in F} \pi_j(p) &\geq \sum_{i\in N} u_i(X_i) - \sum_{j\in F} c_j(Y_j) + [\sum_{j\in F} < Y_j, p > - \sum_{i\in N} < X_i, p >] \\ &\geq \sum_{i\in N} u_i(X_i) - \sum_{j\in F} c_j(Y_j),\end{aligned}$$

since the term $[\sum_{j\in F} < Y_j, p > - \sum_{i\in N} < X_i, p >]$ is nonnegative for a feasible allocation $(X;Y)$. It follows that

$$\sum_{i\in N} v_i(p) + \sum_{j\in F} \pi_j(p) \geq \sum_{i\in N} u_i(X_i) - \sum_{j\in F} c_j(Y_j)$$

for all feasible allocations $(X;Y)$. The lemma follows since an optimal allocation is feasible. □

Lemma 2 provides a characterization for all Walrasian equilibria.

Lemma 2: For $p \in R^l_+$, p is Walrasian if and only if

$$\sum_{i\in N} v_i(p) + \sum_{j\in F} \pi_j(p) = V.$$

Proof: Let $(p^*; X^*; Y^*)$ be a Walrasian equilibrium. Then

$$\begin{aligned}\sum_{i\in N} v_i(p^*) + \sum_{j\in F} \pi_j(p^*) &= \sum_{i\in N} u_i(X_i^*) - \sum_{j\in F} c_j(Y_j^*) \\ &\geq V \qquad \text{by Lemma 1} \\ &\geq \sum_{i\in N} u_i(X_i^*) - \sum_{j\in F} c_j(Y_j^*) \qquad \text{by the definition of } V.\end{aligned}$$

To show the other side. We need to show that there exists an optimal feasible allocation $(X^*; Y^*)$ such that $(p^*; X^*; Y^*)$ is Walrasian whenever

$$\sum_{i\in N} v_i(p^*) + \sum_{j\in F} \pi_j(p^*) = V.$$

Suppose there does not exist any optimal feasible allocation $(X_1, \cdots, X_n; Y_1, \cdots, Y_m)$ such that $X_i \in D_i(p^*)$ and $Y_j \in S_j(p^*)$. Then there exists at least one consumer i or one firm j such that $v_i(p^*) > v_i(X_i, p^*)$ or $\pi_j(p^*) > \pi_j(Y_j, p^*)$ for any optimal feasible allocation $(X;Y)$. Then

$$\begin{aligned}V &= \sum_{i\in N} v_i(p^*) + \sum_{j\in F} \pi_j(p^*) \\ &> \sum_{i\in N} u_i(X_i) - \sum_{j\in F} c_j(Y_j) + [\sum_{j\in F} < Y_j, p^* > - \sum_{i\in N} < X_i, p^* >] \\ &\geq \sum_{i\in N} u_i(X_i) - \sum_{j\in F} c_j(Y_j) \qquad \text{since } (X;Y) \text{ is feasible and optimal} \\ &= V,\end{aligned}$$

a contradiction.

Therefore an optimal feasible allocation $(X^*;Y^*)$ exists such that $X_i^* \in D_i(p^*)$ and $Y_j^* \in S_j(p^*)$ for all $i \in N$ and $j \in F$. We need to show that the market clearing condition

$$\cup_{i\in N} X_i^* = \cup_{j\in F} Y_j^*$$

is satisfied. To see this, note that

$$\begin{aligned} V &= \sum_{i\in N} v_i(p^*) + \sum_{j\in F} \pi_j(p^*) \\ &= \sum_{i\in N} u_i(X_i^*) - \sum_{j\in F} c_j(Y_j^*) + [\sum_{j\in F} < Y_j^*, p^* > - \sum_{i\in N} < X_i^*, p^* >] \\ &= V + [\sum_{j\in F} < Y_j^*, p^* > - \sum_{i\in N} < X_i^*, p^* >]. \end{aligned}$$

Therefore, $\sum_{j\in F} < Y_j^*, p^* >= \sum_{i\in N} < X_i^*, p^* >$. Since $(X^*;Y^*)$ is feasible, we always have that $\cup_{i\in N} X_i^* \subset \cup_{j\in F} Y_j^*$. If $p_a > 0$ for all $a \in \cup_{j\in F} Y_j^*$, then $\sum_{j\in F} < Y_j^*, p^* >= \sum_{i\in N} < X_i^*, p^* >$ implies that $\cup_{i\in N} X_i^* = \cup_{j\in F} Y_j^*$. On the other hand, if there exists some $a^* \in \cup_{j\in F} Y_j^*$ and $a^* \notin \cup_{i\in N} X_i^*$ such that $p_{a^*}^* = 0$, then define $\tilde{Y}_j = Y_j^* \setminus \{a^*\}$, where j is chosen such that $a^* \in Y_j^*$. It follows that

$$\pi_j(\tilde{Y}_j, p^*) > \pi_j(Y_j^*, p^*),$$

by the strictly monotone cost functions. This is a contradiction, since $Y_j^* \in S_j(p^*)$. This shows that

$$\cup_{i\in N} X_i^* = \cup_{j\in F} Y_j^*.$$

Therefore the market clearing condition is satisfied and this completes the proof of the lemma. □

From Lemma 2, we can derive the first and second theorems of welfare economics; also see Gul and Stacchetti (1996a).

Theorem 1 (First Theorem of Welfare Economics): Every Walrasian equilibrium allocation is optimal.

Proof: It directly follows from Lemma 2. □

Theorem 2 (Second Theorem of Welfare Economics): Let $(p;X;Y)$ be a Walrasian equilibrium and $(\bar{X};\bar{Y})$ be an optimal allocation. Then $(p;\bar{X};\bar{Y})$ is a Walrasian equilibrium.

Proof: Use the same argument in the proof of Lemma 2 to show that the market clearing condition is satisfied at $(p; \bar{X}; \bar{Y})$. □

Note that $\cup_{i \in N} X_i$ may be quite different from $\cup_{i \in N} \bar{X}_i$ in Theorem 2. Nonetheless, both $(X; Y)$ and $(\bar{X}; \bar{Y})$ are Walrasian. A useful corollary of Theorem 2 is as follows:

Corollary 1: Let p^* be a Walrasian price vector. Then for all optimal allocations $(X; Y)$, we have that

$$X_i \in D_i(p^*), \forall i \in N$$

and

$$Y_j \in S_j(p^*), \forall j \in F.$$

4 Existence: A Necessary and Sufficient Condition

The following example is first used by Kelso and Crawford (1982) to show that a Walrasian equilibrium may not exist in some job-matching markets. We adopt it to our production economy to show that a Walrasian equilibrium may not exist for some production economies.

Example 2: Consider the following economy with two consumers, j and k. There are three differentiated products, $\Omega = \{1, 2, 3\}$, in the economy. Consumers j and k have utility functions as follows (see Kelso and Crawford (1982)):

$$u_j(\{1\}) = 4, \quad u_j(\{2\}) = 4, \quad u_j(\{3\}) = 4 + \epsilon_1$$

$$u_j(\{1,2\}) = 7 + \epsilon, \quad u_j(\{1,3\}) = 7, \quad u_j(\{2,3\}) = 7$$

$$u_j(\{1,2,3\}) = 9,$$

$$u_j(A) = 0 \quad \text{for any others } \emptyset \neq A \in 3^\Omega$$

$$u_k(\{1\}) = 4 + \epsilon_2, \quad u_k(\{2\}) = 4, \quad u_k(\{3\}) = 4$$

$$u_k(\{1,2\}) = 7, \quad u_k(\{1,3\}) = 7, \quad u_k(\{2,3\}) = 7 + \epsilon$$

$$u_k(\{1,2,3\}) = 9,$$

$$u_k(A) = 0 \quad \text{for any others } \emptyset \neq A \in 3^\Omega$$

where $\epsilon \in [0,1]$ and $\epsilon_1, \epsilon_2 \in [0,3]$.

There are three producers, α, β and γ, each specializes only in producing one unit of their own product. Their cost functions are as follows:

$$c_\alpha(\emptyset) = 0, \quad c_\alpha(\{1\}) = 1, \quad u_\alpha(A) = \infty \quad \text{for all others } \emptyset \neq A \in 3^\Omega$$

$$c_\beta(\emptyset) = 0, \quad c_\beta(\{2\}) = 1, \quad c_\beta(A) = \infty \quad \text{for all others } \emptyset \neq A \in 3^\Omega$$

$$c_\gamma(\emptyset) = 0, \quad c_\gamma(\{3\}) = 1, \quad c_\gamma(A) = \infty \quad \text{for all others } \emptyset \neq A \in 3^\Omega$$

Consider an economy with $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ and $\epsilon = \frac{1}{2}$. There are two optimal allocations

$$(X_j, X_k; Y_\alpha, Y_\beta, Y_\gamma) = (\{1\}, \{2,3\}; \{1\}, \{2\}, \{3\})$$

and

$$(\bar{X}_j, \bar{X}_k; Y_\alpha, Y_\beta, Y_\gamma) = (\{1,2\}, \{3\}; \{1\}, \{2\}, \{3\}).$$

By Corollary 1, a Walrasian equilibrium price vector p must satisfy the following

$$X_j, \bar{X}_j \in D_j(p); \qquad X_k, \bar{X}_k \in D_k(p)$$

and

$$Y_\eta \in S_\eta(p), \text{ where } \eta = \alpha, \beta, \gamma.$$

By $\{1\} \in D_j(p)$, we have that

$$u_j(\{1\}) - p_1 \geq u_j(\{3\}) - p_3$$

which implies that

$$p_3 - p_1 \geq \frac{1}{4}.$$

By $\{3\} \in D_k(p)$, we have that

$$u_k(\{3\}) - p_3 \geq u_k(\{1\}) - p_1$$

which implies that

$$p_1 - p_3 \geq \frac{1}{4}.$$

These two inequalities contradict with each others. Therefore, no Walrasian equilibria exist in the economy with $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ and $\epsilon = \frac{1}{2}$. □

We now consider a production economy such that both consumers and producers are rationed to some degree. Let $\bar{\Omega} \subset 3^\Omega$, which is typically a much smaller set in 3^Ω. We suppose that both consumers and producers can only make their choices of consumption and production in $\bar{\Omega}$. We

can carry all our definitions and results in Sections 2 and 3 over the economy with $\bar{\Omega}$. Note that both Lemmas 1 and 2 are still true when $\bar{V}$ is defined under the restrained $\bar{\Omega}$. But if $\bar{\Omega}$ includes all elements under an optimal allocation of the economy $\mathcal{E}$, then $\bar{V} = V$. Based on this idea, we now obtain a necessary and sufficient condition for the existence of Walrasian equilibrium.

Let $(X^*; Y^*)$ be any optimal allocation in the production economy $\mathcal{E}$ and $\bar{\Omega} = \cup_{j \in F} Y_j^*$. We consider the following linear programming

$$\begin{aligned} \Gamma \quad = \quad & \min_{p \in R_+^l} \min_{x,y} \sum_{i=1}^{i=n} x_i + \sum_{j=1}^{m} y_j \\ s.t. \quad & x_i + < A, p > \ \geq \ u_i(A), \forall A \in 2^{\bar{\Omega}}, \forall i \in N \\ & y_j - < A, p > \ \geq \ -c_j(A), \forall A \in 2^{\bar{\Omega}}, \forall j \in F \\ & x_i, y_j \geq 0, \forall i, j. \end{aligned}$$

The linear programming Γ may be solved with computers.

Theorem 3: A Walrasian equilibrium exists in the production economy $\mathcal{E}$ if and only if $\Gamma = V$.

Proof: Let $(p^*; X^*; Y^*)$ be a Walrasian equilibrium. Since $(X^*; Y^*)$ is optimal by Theorem 1, let $\bar{\Omega} = \cup_{j \in F} Y_j^*$ in Γ. This is without loss of generality by Theorem 2. We show that $(p^*, v(p^*), \pi(p^*))$ is a solution to Γ and $\Gamma = V$. Let $x_i^* = v_i(p^*)$ and $y_j^* = \pi_j(p^*)$. By the equilibrium conditions, both x^* and y^* satisfy the constrains in Γ for p^*. And then, $\Gamma \leq V$. But Lemma 1 shows that $\Gamma \geq V$. This shows that (p^*, x^*, y^*) is a solution to Γ and $\Gamma = V$.

Let (p, x, y) be a solution to Γ such that $\Gamma = V$. We show that $(X^*; Y^*)$ is a collection such that

$$x_i = u_i(X_i^*) - < X_i^*, p >, \forall i \in N,$$

and

$$y_j = < Y_j^*, p > - c_j(Y_j^*), \forall j \in F.$$

Suppose on the contrary that this is not true. Then,

$$\begin{aligned} V \quad = \quad & \Gamma \\ = \quad & \sum_{i=1}^{i=n} x_i + \sum_{j=1}^{m} y_j \\ > \quad & \sum_{i \in N} u_i(X_i^*) - \sum_{j \in F} c_j(Y_j^*) + [\sum_{j \in F} < Y_j^*, p > - \sum_{i \in N} < X_i^*, p >] \end{aligned}$$

$$\begin{aligned} &\geq \sum_{i\in N} u_i(X_i^*) - \sum_{j\in F} c_j(Y_j^*) \\ &= V, \end{aligned}$$

a contradiction. What left is to show that the market clearing condition

$$\cup_{i\in N} X_i^* = \cup_{j\in F} Y_j^*$$

is satisfied. But this can be shown in the same manner as in the proof of Lemma 2. Therefore $(p; X^*; Y^*)$ is a Walrasian equilibrium in $\mathcal{E}$. □

In the production economy there may exist many optimal allocations that may be different from each others in terms of the numbers of copies of each product; see Example 1. But Theorem 3 only needs to check one among all optimal allocations. This simplifies the task in a great manner in comparison with Lemma 2, since 3^Ω contains members with infinite numbers of copies of products. But in the linear programming Γ, $\bar{\Omega}$ is a relatively smaller set and we only need to work with all subsets of $\bar{\Omega}$ rather than all unordered families of elements of Ω.

5 Convergence Theorems

In what follows we first develop some intuitions of the results in this section. Let us consider an economy with a single firm that produces only one product. Firm starts with a very high introductory price of her product that may well create oversupplies of that product. Firm cuts its price as long as oversupplies exist. But firm may cut too much its price that creates overdemands of its product. When there exist overdemands, firm has to raise its price in order to get to the equilibrium. But firm may raise her price that creates oversupplies of her product. Firm may stalk with some cycles. How should firm cut its price such that it eventually arrives at the equilibrium price without cycles? There are many answers to the question. One answer among them that seems natural is that firm should cut its price gradually such that no overdemands are created at each price cut. Since the initial price set by the firm is bounded above and the distance of this price to the equilibrium price is a finite number. It will not take the firm infinitely many price cuts to reach the equilibrium price no matter how small each price cut is. In practice firm may cut its price that is indeed below the equilibrium price. But as long as firm does not cut its price that is below the equilibrium infinitely many times, firm is still able to reach the equilibrium price.

This example with a single firm is the basic intuition behind the convergence result presented in Theorem 4 below. The question is how to formalize the above idea when there are many firms in an economy and their products are interdependent. In what follows we suggest one way how this may be done.

Let $\mathcal{T} = \{0, 1, 2, 3, \cdots, \}$ represent the infinite time horizon. Each t represents a period of time. $\mathcal{E}(t) = ((N(t), u(t)); (F(t), c(t)); (\Omega(t), 3^{\Omega(t)}))$ is a sequence of economies along the time horizon. Let $p_t \in R_+^{|\Omega(t)|}$ be a price vector at time t and $\{p_t\}_{t\in\mathcal{T}}$ be a price process. A *stationary economy* is such that $\mathcal{E}(t) = \mathcal{E}$ for some $\mathcal{E}$ for all t. In what follows we only consider a stationary economy. Our concern is the sufficient conditions under which certain price process $\{p_t\}_{t\in\mathcal{T}}$ converges to a Walrasian equilibrium within finite periods of time for a stationary economy. First for $x, y \in R^l$, $x < y$ means that $x_i \leq y_i$ for all i and $x_i < y_i$ for some i. A price process $\{p_t\}$ is ascending (descending) if $p_{t+1} > p_t$ ($p_{t+1} < p_t$) for each t.

Define

$$\mathcal{O}(p) = \{(A_1, \cdots, A_n; B_1, \cdots, B_m) \mid A_i \in D_i(p), B_j \in D_j(p)\}, \;\; p \in R_+^l.$$

Given $A \in 3^\Omega$, denote $\#A^\omega$ the number of copies of product ω in A. We say that a product $\omega \in \Omega$ is (weakly) *oversupplied* with respect to a collection $(A; B) \in \mathcal{O}(p)$ if

$$\sum_{i\in N} \#A_i^\omega \leq \sum_{j\in F} \#B_j^\omega.$$

A price vector p has the *aggregate oversupplied property* if it induces at least one collection $(A; B) \in \mathcal{O}(p)$ such that every product is oversupplied with respect to $(A; B)$. A price process $\{p_t\}$ has the *aggregate oversupplied property* if p_t has the aggregate oversupplied property for each t.

The aggregate oversupplied property is necessary for a price vector p to be Walrasian. Otherwise, we have

$$\sum_{i\in N} \#A_i^\omega > \sum_{j\in F} \#B_j^\omega$$

for every $(A; B) \in \mathcal{O}(p)$ and every ω. Such a price vector does not admit any Walrasian equilibrium. But the aggregate oversupplied property is not sufficient for p to be Walrasian. It is of interest that this property is sufficient for a descending price process to converge to a Walrasian equilibrium; as shown in Theorem 4.

We start with p_0 that are bounded from above but far higher than the equilibrium prices. For example, we may choose p_0 such that $p_0 = (M + 1, M + 1, \cdots, M + 1)$ at which each consumer is unwilling to buy any commodity bundle. We now consider all possible descending price processes

that converge to Walrasian equilibria within finite periods of time under the aggregate oversupplied property.

Theorem 4: Let $\{p_t\}_{t\in\mathcal{T}}$ be any descending price process that has the aggregate oversupplied property. Then (a) a Walrasian equilibrium exists; (b) there exists a finite time $T^* \in \mathcal{T}$ such that p_{T^*} is Walrasian.

Proof : Fix any t such that p_s is not Walrasian for any time period $s \leq t$. Otherwise, there is nothing to prove. Let $s \in \{t, t+1, \cdots\}$. Define

$$\mathcal{F}(s) = \{(A;B) \in \mathcal{O}(s) \mid \sum_{i\in N} \#A_i^\omega \leq \sum_{j\in F} \#B_j^\omega\}.$$

Also define

$$\mathcal{G}(s) = \sum_{i\in N} v_i(p_s) + \sum_{j\in F} \pi_j(p_s).$$

First we show that

$$\mathcal{G}(s) \geq \mathcal{G}(s+1)$$

for any s. (Note that if there is some finite s such that p_s is Walrasian, then the proof of the theorem is completed. Therefore, the above inequality applies to those time periods s such that p_s is not Walrasian.)

Since $\{p_t\}$ satisfies the aggregate oversupplied property, $\mathcal{F}(s+1)$ is nonempty. Let $(A(s+1); B(s+1)) \in \mathcal{F}(s+1)$. It follows that

$$\sum_{j\in F} < B_j(s+1), p_s - p_{s+1} > - \sum_{i\in N} < A_i(s+1), p_s - p_{s+1} > \ \geq \ 0,$$

since $\{p_s\}$ is descending and

$$\sum_{i\in N} \#A_i^\omega(s+1) \leq \sum_{j\in F} \#B_j^\omega(s+1)$$

for each $\omega \in \Omega$. Then, we have that

$$\begin{aligned}
\mathcal{G}(s) - \mathcal{G}(s+1) &= [\sum_{i\in N} v_i(p_s) - \sum_{i\in N} v_i(p_{s+1})] + [\sum_{j\in F} \pi_j(p_s) - \sum_{j\in F} \pi_j(p_{s+1})] \\
&= [\sum_{i\in N} v_i(p_s) - \sum_{i\in N} v_i(A_i(s+1), p_s) + \sum_{i\in N} < A_i(s+1), p_{s+1} - p_s >] \\
&\quad + [\sum_{j\in F} \pi_j(p_s) - \sum_{j\in F} \pi_j(B_j(s+1), p_s) + \sum_{j\in F} < B_j(s+1), p_s - p_{s+1} >]
\end{aligned}$$

$$
\begin{aligned}
&= \{[\sum_{i\in N} v_i(p_s) - \sum_{i\in N} v_i(A_i(s+1), p_s)] + [\sum_{j\in F} \pi_j(p_s) - \sum_{j\in F} \pi_j(B_j(s+1), p_s)]\} \\
&\quad + [\sum_{i\in N} < A_i(s+1), p_{s+1} - p_s >] + \sum_{j\in F} < B_j(s+1), p_s - p_{s+1} >] \\
&\geq [\sum_{j\in F} < B_j(s+1), p_s - p_{s+1} > - \sum_{i\in N} < A_i(s+1), p_s - p_{s+1} >] \\
&\geq 0.
\end{aligned}
$$

Now define

$$\mathcal{H}_1(s) = \{(A(s); B(s)) \in \mathcal{O}(s) \mid A_i(s) \notin D_i(p_t) \text{ for some } i \in N\},$$

$$\mathcal{H}_2(s) = \{(A(s); B(s)) \in \mathcal{O}(s) \mid B_j(s) \notin S_j(p_t) \text{ for some } j \in F\}$$

and

$$\mathcal{H}(s) = \mathcal{H}_1(s) \cup \mathcal{H}_2(s).$$

$\mathcal{H}(s)$ contains all collections $(A(s); B(s)) \in \mathcal{O}(s)$ such that $A_i(s) \notin D_i(p_t)$ for some customer $i \in N$ or $B_j(s) \notin S_j(p_t)$ for some firm $j \in F$.

We show that there exists a finite $s \in \{t+1, t+2, \cdots\}$ such that

$$\mathcal{H}(s) \cap \mathcal{F}(s) \neq \emptyset.$$

Suppose on the contrary that $\mathcal{H}(s) \cap \mathcal{F}(s) = \emptyset$ for every $s \in \{t+1, t+2, \cdots\}$. Then $A_i(s) \in D_i(p_t)$ for all $i \in N$ and $B_j(s) \in S_j(p_t)$ for all $j \in F$ for every $(A(s); B(s)) \in \mathcal{F}(s)$ for every s. Pick up any $(A(s); B(s)) \in \mathcal{F}(s)$ and any j such that $B_j(t) = B_j(s) \neq \emptyset$. If p_s is Walrasian for some finite s, there is nothing to prove. Suppose that p_s is not Walrasian for every s. Then let $q \in R^l_+$ be a price vector such that

$$< B_j(t), q > - c_j(B_j(t)) < -c_j(\{\emptyset\}), \forall B_j(t) \in S_j(p_t).$$

$q \geq 0$ since c_j is strictly monotone. Thus $B_j(t) \notin S_j(q)$ for all $B_j(t) \in S_j(p_t)$. Since $\{p_t\}$ is descending and p_s is not Walrasian for every s, there exists a finite s^* such that $p_{s^*} \leq q$ and then $B_j(t) \notin S_j(p_{s^*})$ for all $B_j(p_t) \in S_j(p_t)$. This is a contradiction.

We now show that there exist a finite T and $\delta > 0$ such that

$$\mathcal{G}(t) \geq \cdots \geq \mathcal{G}(t+T) + \delta.$$

We know from above that

$$\mathcal{G}(t) \geq \cdots \geq \mathcal{G}(t+T).$$

Note that there exists some $\epsilon > 0$ such that $\pi_j(p) \geq \pi_j(B,p) + \epsilon$ for any $B \notin S_j(p)$ and $v_i(p) \geq v_i(A,p) + \epsilon$ for any $A \notin D_i(p)$.

Since there exists a finite T such that $\mathcal{H}(t+T) \cap \mathcal{F}(t+T) \neq \emptyset$, let $(A;B)$ be a collection in $\mathcal{H}(t+T) \cap \mathcal{F}(t+T)$. Then,

$$\sum_{j \in F} < B_j, p_t - p_{t+T} > - \sum_{i \in N} < A_i, p_t - p_{t+T} > \ \geq \ 0,$$

and there exists $\delta > 0$ such that (see $\mathcal{G}(s) - \mathcal{G}(s+1)$ above)

$$\begin{aligned}
\mathcal{G}(t) - \mathcal{G}(t+T) &= \{[\sum_{i \in N} v_i(p_t) - \sum_{i \in N} v_i(A_i, p_t)] + [\sum_{j \in F} \pi_j(p_t) - \sum_{j \in F} \pi_j(B_j, p_t)]\} \\
&\quad + [\sum_{i \in N} < A_i, p_{t+T} - p_t >] + \sum_{j \in F} < B_j, p_t - p_{t+T} >] \\
&\geq \delta + [\sum_{j \in F} < B_j, p_t - p_{t+T} > - \sum_{i \in N} < A_i, p_t - p_{t+T} >] \\
&\geq \delta.
\end{aligned}$$

In summary, as the price process $\{p_t\}$ continues and no Walrasian equilibrium appears, there exist some finites $T_1, T_2, \cdots$ and $\delta_1 > 0, \delta_2 > 0, \cdots$ such that

$$\mathcal{G}(t) \geq \cdots \geq \delta_1 + \mathcal{G}(t+T_1) \geq \cdots \geq \delta_1 + \delta_2 + \mathcal{G}(t+T_1+T_2) \geq \cdots.$$

Since $\mathcal{G}(s)$ is bounded below by V according to Lemma 1, the above process cannot last forever. Therefore, there must exist a finite T^* such that

$$\mathcal{G}(T^*) = V.$$

Then, by Lemma 2, p_{T^*} must be Walrasian. □

We say that a product $\omega \in \Omega$ is (weakly) *overdemanded* with respect to a collection $(A;B) \in \mathcal{O}(p)$ if

$$\sum_{i \in N} \#A_i^\omega \geq \sum_{j \in F} \#B_j^\omega.$$

A price vector p has the *aggregate overdemanded property* if it induces at least one collection $(A;B) \in \mathcal{O}(p)$ such that every product is overdemanded with respect to $(A;B)$. A price process $\{p_t\}$ has the *aggregate overdemanded property* if p_t has the aggregate overdemanded property for each t.

It is necessary for a price vector p to satisfy the aggregate overdemanded property in order for p to be Walrasian. Otherwise, we have

$$\sum_{i \in N} \# A_i^\omega < \sum_{j \in F} \# B_j^\omega$$

for all $(A; B) \in \mathcal{O}(p)$ and all $\omega \in \Omega$. Such a price vector p cannot be Walrasian. Again, the aggregate overdemanded property is not a sufficient condition for p to be Walrasian. But this property is a sufficient condition for an ascending price process to converge to a Walrasian equilibrium; as shown in Theorem 5 below. The proof of Theorem 5 is immediate by following that of Theorem 4 in a symmetric manner and is thus omitted.

Theorem 5: Let $p_0 = 0 \in R_+^l$ and $\{p_t\}_{t \in \mathcal{T}}$ be any ascending price process that has the aggregate overdemanded property. Then (a) a Walrasian equilibrium exists; (b) there exists a finite time $T^* \in \mathcal{T}$ such that p_{T^*} is Walrasian.

Two useful corollaries of Theorems 4 and 5 are as follows:

Corollary 2: Let $\{p_t\}_{t \in \mathcal{T}}$ be any descending price process and $\{q_t\}$ be a subsequence of $\{p_t\}$ that has the aggregate oversupplied property. Then (a) a Walrasian equilibrium exists; (b) there exists a finite time $T^* \in \mathcal{T}$ such that q_{T^*} is Walrasian.

Corollary 3: Let $\{p_t\}_{t \in \mathcal{T}}$ be any ascending price process and $\{q_t\}$ be a subsequence of $\{p_t\}$ that has the aggregate overdemanded property. Then (a) a Walrasian equilibrium exists; (b) there exists a finite time $T^* \in \mathcal{T}$ such that q_{T^*} is Walrasian.

One may think that the conditions in Theorems 4 and 5 are strong. In fact once an economy has a Walrasian equilibrium, there exist at least one descending price process that satisfies the condition in Theorem 4 and one ascending price process that satisfies the condition in Theorem 5. The following example shows that there may exist many price processes that satisfy the condition in Theorem 4 for an economy with a Walrasian equilibrium. Our results in Theorems 4 and 5 show more than this. Once an economy has such a price process that satisfies the condition in Theorem 4 or 5, it must have a Walrasian equilibrium.

Consider the economy in Example 1 with parameters, $\epsilon_1 = 1.75$ and $\epsilon_2 = \epsilon = .5$. Thus,

$V = 9.75$. The price vector $(3, 2.5, 3.5)$ is Walrasian, at which

$$D_j(p) = (\{3\}), \qquad D_k(p) = (\{1\}, \{2\}, \{1,2\}, \{2,3\})$$

$$S_\alpha(p) = (\{1\}), \quad S_\beta(p) = (\{2\}), \quad S_\gamma(p) = (\{3\}).$$

We start with a price vector $p_0 = (5, 5, 5)$.

t	p_t	D_j	D_k	$\mathcal{G}(t)$
0	$(5,5,5)$	$(\{3\})$	$(\{\emptyset\})$	12.75
1	$(4,4,5)$	$(\{3\})$	$(\{1\})$	11.25
2	$(4,3,4)$	$(\{3\})$	$(\{2\})$	10.75
3	$(3,3,4)$	$(\{3\})$	$(\{1\})$	10.25
4	$(3,2.75,4)$	$(\{3\},\{1,2\})$	$(\{1\})$	10
5	$(3,2.75,3.5)$	$(\{3\})$	$(\{1\})$	10
6	$(3,2.5,3.5)$	$(\{3\})$	$(\{1\},\{2\},\{1,2\},\{2,3\})$	9.75

In the price process $\{p_t\}$ above,

$$S_\alpha(p_t) = (\{1\}), S_\beta(p_t) = (\{2\}), S_\gamma(p_t) = (\{3\})$$

for each t. The price process $\{p_t\}$ satisfies the conditions in Theorem 4 and it converges to a Walrasian equilibrium. The price process $\{p_t\}$ seems reasonable. Once a firm finds out that her product is not demanded at a period of time, she cuts her price next period while those firms whose products are demanded in the prior period will stay put. Note that the price of product 3 from $t = 4$ to $t = 5$ is lower because firm γ directly competes with α and β. Of course this price process is quite artificial because we know where a Walrasian equilibrium is. One can design many "meaningful" price processes that satisfy the conditions in Theorems 4 and 5 without any difficulties.

6 An Oligopoly Economy with Constant Marginal Costs

This section presents a special economy in which each product is produced by a monopoly with constant marginal costs. Precisely we identify $\Omega = F$. For each $j \in F$, $c_j(\{\emptyset\}) = 0, c_j(J) = mc_j \# J$ for all $J \in 3^{\{j\}}$ and $c_j(J) = \infty$ for any other $J \in 3^F$. Denote $MC = (mc_1, mc_2, \cdots, mc_m)$. In the economy $\tilde{\mathcal{E}}$ there exists a unique Walrasian equilibrium price vector $p = MC$, i.e., each Walrasian price p_j equals the marginal cost mc_j of firm j.

Observation: Let $(X;Y)$ be an optimal allocation in the economy $\tilde{\mathcal{E}}$. Then each X_i is a solution to the problem

$$\max\{u_i(A)- < A, MC > | \ A \in 3^{\Omega}\}.$$

Proof: Suppose on the contrary that there exists some i such that X_i is not a solution to the problem. Let A^* be a solution to the problem

$$\max\{u_i(A)- < A, MC > | \ A \in 3^{\Omega}\}.$$

Then we have

$$u_i(A^*)- < A^*, MC > > \ u_i(X_i)- < X_i, MC > .$$

Let $(Y_1^*, \cdots, Y_m^*)$ be a collection such that

$$\cup_{j\in F} Y_j^* = [\cup_{j\in F} Y_j] \cup A^* \setminus X_i$$

and $Y_j^* \in 3^{\{j\}}$ for each $j \in F$. Then $(X_1, \cdots, X_{i-1}, A^*, X_{i+1}, \cdots, X_n; Y_1^*, \cdots, Y_m^*)$ is an allocation. Define

$$V^* = \sum_{i'\in N\setminus\{i\}} u_{i'}(X_{i'}) + u_i(A^*) - \sum_{j\in F} c_j(Y_j^*).$$

Then

$$\begin{aligned} V^* - V &= u_i(A^*) - u_i(X_i) + \sum_{j\in F} c_j(Y_j) - \sum_{j\in F} c_j(Y_j^*) \\ &= u_i(A^*) - u_i(X_i) + < X_i, MC > - < A^*, MC > \\ &> 0. \end{aligned}$$

Thus $(X;Y)$ is not optimal, a contradiction. □

It follows from the above observation that MC is a Walrasian price vector. Note that the market clearing condition is satisfied for all optimal allocations in the economy $\tilde{\mathcal{E}}$. The fact that MC is the unique Walrasian equilibrium price vector follows from that firm j will supply infinitely many copies of her product whenever the price of her product exceeds mc_j and she supplies zero whenever the price of her product is below mc_j. None of these cases has a Walrasian equilibrium.

In such an economy suppose that each firm j starts with a price of her product that is higher than the marginal cost mc_j. Since as long as the price p_j is higher than mc_j, firm j is willing to produce infinitely many copies of her product, the condition in Theorem 4 is always satisfied whenever $p_t \geq MC$ for each t. Thus any descending price process $\{p_t\}$ such that $p_t \geq MC$ for each

t will converge to the Walrasian equilibrium within finite periods of time. Certainly, when the cost functions are not so simple, things become more complicated; as shown in Section 5. This example also shows that the Walrasian (competitive) equilibrium may be observed though each firm is a monopoly in her own product.

7 Some Discussions

The equilibria in a production economy will be disturbed if the changes in the underlying economy change the balance of aggregate demand and supply. For example, an increase in the costs in production may change the production plan of firms and the "decrease" in the aggregate supply may create overdemanded products for an economy. Overdemanded products may eventually bid up the prices of products and induce inflation. But if the increase in the costs in production is not so dramatic to change the balance of aggregate demand and supply, then the price level in an economy may stay the same. But how do we exactly know when the changes in the costs of production are dramatic to change the equilibrium of an economy? The impact of the changes lies on the social surplus function, V, and the total consumer surpluses and profits. If the increases in the costs of production do not change the equation in Lemma 2, then the equilibrium price vector will be the same before and after the changes, though Walrasian allocations may be quite different. But if the changes in the costs change the equation in Lemma 2, the economy will likely move to a new price level at which some products must experience higher prices. This price adjustment process will not take infinitely periods of time for those ascending price processes with the aggregate overdemanded property. Similarly, if consumers have changes in their preferences, then the equilibrium price level may or may not be disturbed. There are some others situations such that the changes of the underlying economy may result in a situation without an equilibrium. If such a situation happens, the adjustment of the prices in the economy is going to end up with a situation in which there always exist some overdemanded or oversupplied products. The price adjustment in the economy will likely experience a complex business cycle. The market system will fail to allocate the resources efficiently. How does an economy behave in such a nonequilibrium situation is unknown at this point. The other untouched issue in this paper is how to embody the uncertainty into the model. Such a model will be quite useful to capture some aspects of the stock or bond market since all bonds and stocks are indivisible assets in nature. They are sold or bought in units (one share of stocks or one $1000 face value of bonds).

We follow the auction and the bargaining literature and the current literature with indivisible commodities to assume that consumers' utility functions do not have income effects. This restrictive feature of the model may make it less applicable in many situations of economic interests. An interesting extension is how to embody the income effects into the model and study their impacts on the existence and convergence results. Some of these issues are left for the future studies.

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