

Ma, Jinpeng

**Working Paper**

## Competitive Equilibrium with Indivisibilities

Working Paper, No. 1998-09

**Provided in Cooperation with:**

Department of Economics, Rutgers University

*Suggested Citation:* Ma, Jinpeng (1998) : Competitive Equilibrium with Indivisibilities, Working Paper, No. 1998-09, Rutgers University, Department of Economics, New Brunswick, NJ

This Version is available at:

<https://hdl.handle.net/10419/94341>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# Competitive Equilibrium with Indivisibilities\*

Jinpeng Ma<sup>†</sup>

Rutgers University

Revised January 25, 1998

## Abstract

This paper studies an exchange economy with a finite number of agents in which each agent is initially endowed with a finite number of (personalized) indivisible commodities. We observe that the equivalence theorem of the core and the competitive equilibrium may not hold for this economy when the coalitional form game is generated in the standard manner. We provide an alternative definition of the coalitional form game to resolve this problem so that the balancedness of the new defined game provides a useful necessary and sufficient condition for the existence of competitive equilibrium for the original economy.

We also observe that the nice strategy proof property of the minimum competitive price mechanism in the assignment problem and the Vickrey auction model does not carry over to the above economy. We show that examples of exchange economies exist for which no competitive price mechanism is individually (coalitionally) strategy proof.

*JEL classification* numbers: D41, D44, D50

---

\*I thank Sushil Bikhchandani, Vince Crawford and editor Karl Shell for helpful comments and suggestions. I am grateful to an associate editor for many fruitful comments that greatly improve the exposition of the paper. Of course, any remaining errors are mine.

<sup>†</sup>Department of Economics, Rutgers University, Camden, NJ 08102, jinpeng@crab.rutgers.edu

# 1 Introduction

We consider an exchange economy with a finite number of agents. Each agent is initially endowed with a finite number of indivisible (homogeneous or heterogeneous) commodities and may consume as many items as he wishes. An agent's utility function is a set function over the set of all commodity bundles. This exchange economy is fairly broad. Some related examples of it include the exchange economy in Bikhchandani and Mamer [1] and Gul and Stacchetti [7], the assignment problem (Koopmans and Beckman [12], Shapley and Shubik [20]) and the auction model in Vickrey [21]. It is also analogous to the job-matching market in Kelso and Crawford [11]<sup>1</sup>.

It is known that the assignment problem has a competitive equilibrium.<sup>2</sup> Indeed Shapley and Shubik [20] defined a market game for the assignment problem and showed that the market game is totally balanced and thus has a nonempty core. It follows from the core equivalence theorem that a competitive equilibrium exists in the assignment problem. This existence result depends on the assumption that each buyer consumes at most one unit. If buyers can consume as many as they wish, then a competitive equilibrium may not exist; see Bikhchandani and Mamer [1], Kelso and Crawford [11] and Section 3 for examples. This motivates a natural question: Under what circumstance does a competitive equilibrium exist under the general situation? Kelso and Crawford [11] studied a job-matching market and discovered that if firm's utility function satisfies their *gross substitutes* condition, their matching model has a nonempty core and thus has a competitive equilibrium since a core matching is also competitive (Kelso and Crawford [11, pp.1487 & 1502]). Bikhchandani and Mamer [1] used a different approach and obtained a necessary and sufficient condition for the existence of competitive equilibrium. In their approach, they designed two linear programmings and showed that a competitive equilibrium exists if and only if the two linear programmings have solutions in common. Lately, Gul and Stacchetti [7] found two new sufficient conditions, the *no complementarities* and the *single improvement* property, on buyer's utility function for the existence of competitive equilibrium. They showed that their two new conditions are equivalent to the gross substitutes condition in Kelso and Crawford [11] and then invoked the nonempty core theorem in Kelso and Crawford [11] to obtain their existence theorem.

This paper uses the core approach in Shapley and Shubik [20] to study the same existence issue. We use the economy to generate a coalitional form game and want to show that the balancedness of the game provides a useful necessary and sufficient condition for the existence. Typically the number of players in a coalitional form game generated by an economy is the same as the number

---

<sup>1</sup>But it should be aware that a worker's utility function in Kelso and Crawford [11] is more complicated and it depends on not only the received salary but also the name of a firm with whom he is matched. This is quite different from the exchange economy of this paper in which agents are not concerned with the names of the owners of the commodities.

<sup>2</sup>Gale [5], Kaneko [8,9], Kaneko and Yamamoto [10] and Quinzii [15] obtained several existence theorems for quite general situations with indivisible goods.

of agents in the economy. But we observe that when the coalitional form game is generated in this standard manner, its core does not coincide with the set of competitive payoffs and the balancedness does not provide sufficient information on the existence of competitive equilibrium for the original economy; as shown in Example 1. To resolve the problem, we provide an alternative way to generate a coalitional form game. The approach is as follows. Suppose that an economy has one agent  $i$  with no indivisible commodities and one agent  $j$  with two indivisible commodities, 1 and 2. Given such an economy, we consider a bilateral exchange economy with four “agents”,  $i, j, 1$  and  $2$ , such that agents  $i$  and  $j$  have the same utility functions as they do in the original economy but both have no indivisible commodities (even though agent  $j$  owns two commodities in the original economy), and (commodity) “agents” 1 and 2 have zero utility functions but own commodities 1 and 2 respectively. Then we use this bilateral exchange economy to generate a coalitional form game for the original exchange economy. It follows from the argument made in Kelso and Crawford [11, pp.1487 & 1502]) that the core equivalence theorem holds for this bilateral exchange economy, i.e., the core of the coalitional form game coincides with the set of competitive equilibrium payoffs. Our observation is that a competitive equilibrium exists in the bilateral exchange economy if and only if it exists in the original exchange economy. Therefore a competitive equilibrium in the original economy exists if and only if the generated game on  $\{i, j, 1, 2\}$  is balanced, since the balancedness of the game is the necessary and sufficient condition for it to have a nonempty core (Bondareva [2], Shapley [18]).

The above idea is in fact not completely new and it has been implicitly used in Kelso and Crawford [11] before. In their one-sided “matching” market, they start with the coalitional form game and construct a job-matching market by adding dummy buyers, each with an identical utility function. Here we consider an exchange economy with heterogeneous utility functions and construct a bilateral exchange economy to generate a game by adding dummy commodity sellers. This approach is quite useful because it delivers information on existence or nonexistence of a competitive equilibrium at the same time. Moreover, we can say exactly what are the competitive prices in the original exchange economy. They are nothing but the core payoffs of the coalitional form game “received” by those commodity “sellers”.

Our second concern is the incentive aspects of competitive price mechanisms. A mechanism is individually (coalitionally) strategy proof if it is always the best for each agent (each agent in each coalition) to reveal his true information. It is well-known that the minimum competitive price mechanism in the auction model in Vickrey [21] exists and it is individually strategy proof. Demange [3] and Leonard [13] studied the assignment problem and independently showed that the minimum competitive price mechanism is individually strategy proof. Demange and Gale [6] studied a generalized assignment problem and showed that the minimum competitive price mechanism exists and it is individually strategy proof as well. The minimum competitive price mechanism is coalitionally strategy proof in these models if side payments are not allowed. These

three models and their outcomes are bilateral in nature. One may wonder if this bilateral feature is the source of the nice incentive property of the minimum competitive price mechanism. We will show that this is not the case. Indeed we use a bilateral exchange economy and show that the minimum competitive price mechanism exists but it is not immune to the misrepresentation of the true information by individuals or coalitions. In fact we show that examples of exchange economies exist for which no competitive price mechanism is individually or coalitionally strategy proof. Therefore, our result reveals that it is the assumption of unit demand in the assignment problem and the Vickrey auction model that contributes to the strategy proof property of the minimum competitive price mechanism.<sup>3</sup> The intuition is as follows: An agent in the current model has the opportunity to misrepresent his utility function to undercut the commodity bundles in his demand correspondence. This type of misrepresentation decreases competition in the demand side and thus lowers the competitive prices for his consumption bundle. Since coalitional misrepresentation does not need any side payments in the current model, our result may also provide a new insight for the formation of bidder rings, a very fact in the auction practice which is often interpreted as a by-product of side payments or of repeated auctions.

This paper is structured as follows. Section 2 defines the exchange economy. Section 3 proves the main result on the existence of competitive equilibrium. Section 4 studies the incentive properties of competitive price mechanisms.

## 2 The Exchange Economy

Let  $N = \{1, 2, \dots, n\}$  denote the set of agents. Each agent  $i \in N$  is initially endowed with a finite number of indivisible commodities  $\Omega_i = \{\omega_{i1}, \dots, \omega_{ij}, \dots, \omega_{ik_i}\}$ , where  $k_i \in \{0, 1, 2, \dots\}$  is a finite integer. Thus some agents may not have initial physical endowments  $k_i = 0$  and some agents may have units of them  $k_i \geq 1$ . Let  $\Omega = \cup_{i=1}^n \cup_{j=1}^{k_i} \omega_{ij}$  denote the set of all commodities in the economy and  $2^\Omega$  denote the set of all commodity bundles. For an agent  $i \in N$ , his utility function  $u_i$  is a set function  $u_i : 2^\Omega \rightarrow R$  satisfying  $u_i(\emptyset) = 0$ . Henceforth we assume that agents' utility functions are weakly monotone.<sup>4</sup> Following Bikhchandani and Mamer [1] and Gul and Stacchetti [7], we assume that each agent  $i$  is initially endowed with wealth  $W_i \geq u_i(\Omega)$  that enables him to buy any commodity bundle  $A \subset \Omega$ . We denote this economy by  $\mathcal{E}$ .<sup>5</sup>

---

<sup>3</sup>Vickrey noted its importance for the incentive property of his second-price auction mechanism to hold. However, he did not provide evidence how buyers may misrepresent the mechanism if it is not satisfied. The auction literature thereafter often considers the situation that there is one item for sale.

<sup>4</sup>A utility function  $u$  is weakly monotone if for all  $A, B \in \Omega$  such that  $A \subset B$ ,  $u(A) \leq u(B)$ . Free disposal is a sufficient condition for this assumption.

<sup>5</sup>The NBA labor market may provide a good example for this economy if players only care about money. In the NBA labor market a player may be owned by a team or by himself, and a team typically owns many players. In this labor market, a player is a seller but a team may be a seller or a buyer or both. The other example of this economy is the secondary sale market for the spectrum licenses in which firms may trade their licenses after they obtain the

A feasible allocation  $Y$  in the economy  $\mathcal{E}$  is a partition  $(Y(1), \dots, Y(n))$  of  $\Omega$ , where agent  $i$  is allocated the commodity bundle  $Y(i)$ . We introduce some notation from Gul and Stacchetti [7] below. Let  $T = \{1, 2, \dots, t\}$  be a set and  $x \in R^t$  be a vector, define

$$\langle A, x \rangle = \sum_{a \in A} x_a, \text{ where } A \subset T.$$

Given a price vector  $P \in R_+^{|\Omega|}$ , agent  $i$ 's trading profit function  $v_i : 2^\Omega \times R_+^{|\Omega|} \rightarrow R$  is defined by

$$v_i(A, P) = u_i(A) + \langle \Omega_i, P \rangle - \langle A, P \rangle$$

and his demand correspondence  $D_i : R_+^{|\Omega|} \rightarrow 2^\Omega$  is defined by

$$D_i(P) = \{A \subset \Omega : v_i(A, P) \geq v_i(B, P), \forall B \subset \Omega\}.$$

A pair  $(Y, P)$  of a feasible allocation  $Y$  and a price vector  $P$  is a competitive (Walrasian) equilibrium if  $Y(i) \in D_i(P)$  for all agents  $i \in N$ . At a competitive equilibrium, each agent obtains the greatest trading profit.

Define  $v_i(P) = u_i(A) + \langle \Omega_i, P \rangle - \langle A, P \rangle$  for  $A \in D_i(P)$  and  $v(P) = (v_1(P), \dots, v_n(P))$ .

Let

$$W = \{(P, v(P)) \in R_+^{|\Omega|} \times R^{|\Omega|} : P \text{ is competitive}\}$$

denote the set of all competitive equilibrium payoffs.

### 3 The Existence of Competitive Equilibrium

An economy can be used to generate a coalitional form game. Typically the set of agents in the coalitional form game is the same as in the economy. In many economic situations the equivalence theorem of the core and the competitive equilibria holds, i.e., the core of the coalitional form game coincides with the set of competitive equilibrium payoffs. Therefore a sufficient condition for the nonempty core provides a useful condition for the existence of competitive equilibrium, and vice versa. But we observe that the equivalence theorem fails in the bilateral exchange economy in Bikhchandani and Mamer [1] and Gul and Stacchetti [7] and in the exchange economy  $\mathcal{E}$  in Section 2 when the coalitional form game is generated in the standard manner.

**Example 1** Consider the following economy with three agents,  $i$ ,  $j$  and  $k$ . Agent  $i$  is endowed with three objects, 1, 2 and 3, and has zero utility over any bundle of the three objects. Agents  $j$  and  $k$  have utility functions as follows (see Kelso and Crawford [11]):

---

licenses from the auction; see McMillan [14] for an example.

$$\begin{aligned}
u_j(\{1\}) &= 4, & u_j(\{2\}) &= 4, & u_j(\{3\}) &= 4 + \epsilon_1 \\
u_j(\{1, 2\}) &= 7 + \epsilon, & u_j(\{1, 3\}) &= 7, & u_j(\{2, 3\}) &= 7 \\
u_j(\{1, 2, 3\}) &= 9 \\
u_k(\{1\}) &= 4 + \epsilon_2, & u_k(\{2\}) &= 4, & u_k(\{3\}) &= 4 \\
u_k(\{1, 2\}) &= 7, & u_k(\{1, 3\}) &= 7, & u_k(\{2, 3\}) &= 7 + \epsilon \\
u_k(\{1, 2, 3\}) &= 9
\end{aligned}$$

where  $\epsilon \in [0, 1]$  and  $\epsilon_1, \epsilon_2 \in [0, 3]$ . Clearly, these utility functions are weakly monotone.

We now define the coalitional form game  $w$  on  $\{i, j, k\}$  in the standard manner. Then we have that

$$w(\{i\}) = w(\{j\}) = w(\{k\}) = w(\{j, k\}) = 0, \quad w(\{i, j\}) = w(\{i, k\}) = 9$$

and

$$w(\{i, j, k\}) = \begin{cases} 11 + \epsilon & 0 \leq \epsilon_1 \leq \epsilon, 0 \leq \epsilon_2 \leq \epsilon \\ 11 + \epsilon_1 & \epsilon_1 \geq \epsilon_2, \epsilon_1 \geq \epsilon \\ 11 + \epsilon_2 & \epsilon_2 \geq \epsilon_1, \epsilon_2 \geq \epsilon. \end{cases}$$

The core  $C(w)$  is nonempty since  $(w(\{i, j, k\}) - 2, 1, 1)$  is in  $C(w)$  for any  $\epsilon, \epsilon_1$  and  $\epsilon_2$ . But we knew from Kelso and Crawford [11] that the economy with parameters  $\epsilon_1 = \epsilon_2 = \frac{1}{4}$  and  $\epsilon = \frac{1}{2}$  does not have a competitive equilibrium. Therefore, this example shows that the core equivalence theorem fails for the coalitional form game  $w$ . The balancedness condition of the game  $w$  is not sufficient for the existence of competitive equilibrium. *Q.E.D.*

To resolve the problem, we construct from the original exchange economy  $\mathcal{E}$  a bilateral one  $\tilde{\mathcal{E}}$  such that  $N$  and  $\Omega$  are the two disjoint sets of agents in  $\tilde{\mathcal{E}}$  such that each agent  $i$  in  $N$  has the same utility functions as he does in  $\mathcal{E}$  but with no initial endowments and each agent  $\omega \in \Omega$  owns one single indivisible object, namely the object itself, but she has zero utility functions. We use the economy  $\tilde{\mathcal{E}}$  to generate a coalitional form game  $V$  and make use of the core of this game to study the existence of competitive equilibrium in the economy  $\mathcal{E}$ .

**Definition** A feasible allocation in  $\tilde{\mathcal{E}}$  is a map  $X : \Omega \cup N \rightarrow 2^{\Omega \cup N}$  such that (i) for all  $\omega \in \Omega$ ,  $X(\omega) \in N \cup \{\omega\}$ ; (ii) for all  $\omega \in \Omega$  such that  $X(\omega) \not\subseteq N$ ,  $X(\omega) = \omega$ ; (iii) for all  $i \in N$ ,  $X(i) \subset \Omega$ ; (iv) for all  $\omega \in \Omega$  and  $i \in N$ ,  $X(\omega) = i$  if and only if  $\omega \in X(i)$ .

Let  $T(\Omega, N)$  denote the set of all feasible allocations in  $\tilde{\mathcal{E}}$ . Given a subset  $S \subset N \cup \Omega$ , let  $T(S \cap \Omega, S \cap N)$  denote the set of all feasible allocations for the coalition  $S$ . Also let  $\tilde{D}_i(P)$  denote the demand correspondence in the economy  $\tilde{\mathcal{E}}$ . Thus a pair  $(X, P)$  of a feasible allocation  $X$  and a

price vector  $P$  is a competitive equilibrium if (i)  $P \geq 0$ ; (ii)  $P_\omega = 0$  for all  $\omega$  such that  $X(\omega) = \omega$ ; and (iii)  $X(i) \in \tilde{D}_i(P)$  for all  $i \in N$ .

Let

$$\tilde{W} = \{(P, v(P)) \in R_+^{|\Omega|} \times R_+^{|N|} : P \text{ is competitive in } \tilde{\mathcal{E}}\}$$

denote the set of competitive equilibrium payoffs in the economy  $\tilde{\mathcal{E}}$ , where  $v_i(P) = u_i(A) - \langle A, P \rangle$  for  $A \in \tilde{D}_i(P)$ .

We now use the economy  $\tilde{\mathcal{E}}$  to generate a coalitional form game  $V$ . For  $S \subset \Omega \cup N$ , define

$$V(S) = \max_{X_S \in T(S \cap \Omega, S \cap N)} \sum_{i \in S \cap N} u_i(X_S(i)),$$

satisfying  $V(\emptyset) = 0$ .  $V(S)$  is the maximum amount of utilities generated by the coalition  $S$ . This game  $V$  is a natural extension of the coalitional form game in Shapley and Shubik [20].

The core  $C(V)$  of the game  $V$  consists of all payoff vectors  $(x, y) \in R^{|\Omega|} \times R^{|N|}$  such that

$$\begin{aligned} \langle \Omega, x \rangle + \langle N, y \rangle &= V(\Omega \cup N), \\ \langle S \cap \Omega, x \rangle + \langle S \cap N, y \rangle &\geq V(S) \quad \text{for all } S \subset \Omega \cup N. \end{aligned}$$

Our observation is quite simple. First, it follows from Kelso and Crawford [11, pp.1487 & 1502] that the core equivalence theorem holds in the economy  $\tilde{\mathcal{E}}$  when the coalitional form game  $V$  is defined on  $N \cup \Omega$  as above. Second, we show that a competitive equilibrium exists in  $\tilde{\mathcal{E}}$  if and only if it exists in  $\mathcal{E}$ . Finally, it follows from Bondareva [2] and Shapley [18] that the core  $C(V)$  is nonempty if and only if the game  $V$  is balanced.

**Theorem 1** *Assume that  $u_i$  is weakly monotone for all  $i \in N$ . Then a competitive equilibrium exists in  $\mathcal{E}$  if and only if  $V$  is balanced.*

**Proof.** In fact Step 1 below follows from Kelso and Crawford [11, pp.1487 & 1502]. Here we provide a proof for the sake of completeness.

**Step1.**  $C(V) = \tilde{W}$ .

Let  $(x, y) \in C(V)$ . It follows from the definition of the core  $C(V)$  that there exists an optimal allocation  $Y$  in the economy  $\tilde{\mathcal{E}}$  such that

$$\sum_{i \in N} [y_i + \langle Y(i), x \rangle] = \sum_{i \in N} V(i \cup Y(i)) = V(\Omega \cup N).$$

It follows from the core that for all  $A \subset \Omega$ ,

$$y_i + \langle A, x \rangle \geq u_i(A).$$



In particular,

$$y_i + \langle Y(i), x \rangle \geq u_i(Y(i)) = V(i \cup Y(i))$$

for all  $i \in N$ . Therefore,

$$y_i + \langle Y(i), x \rangle = V(i \cup Y(i))$$

for all  $i \in N$ . Clearly  $x \geq 0$ . It follows that  $(Y, x)$  is a competitive equilibrium in  $\tilde{\mathcal{E}}$ .

Let  $(P, v(P)) \in \tilde{W}$ . We show that  $(P, v(P))$  is in the core  $C(V)$ . Clearly,

$$\langle \Omega, P \rangle + \langle N, v(P) \rangle = V(\Omega \cup N)$$

since any competitive allocation  $Y$  must be optimal. Now suppose that there exists a coalition  $S \subset \Omega \cup N$  such that

$$\langle S \cap \Omega, P \rangle + \langle S \cap N, v(P) \rangle < V(S).$$

It follows that  $\exists i \in S \cap N$  and  $\exists A \subset S \cap \Omega$  such that

$$\langle A, P \rangle + v_i(P) < V(i \cup A) = u_i(A).$$

This implies that  $v_i(P) < u_i(A) - \langle A, P \rangle$ , contradicting  $P$  is competitive.

**Step 2.**  $\tilde{W} \neq \emptyset$  iff  $W \neq \emptyset$ .

First note that every competitive price vector in the economies  $\mathcal{E}$  and  $\tilde{\mathcal{E}}$  is nonnegative. This follows from the assumption that agents have weakly monotone utility functions. Let  $S \subset \Omega$  be any commodity bundle. Then for all agents  $i \in N$ ,

$$(*) \quad u_i(S) + \langle \Omega_i, P \rangle - \langle S, P \rangle \geq u_i(C) + \langle \Omega_i, P \rangle - \langle C, P \rangle$$

if and only if

$$u_i(S) - \langle S, P \rangle \geq u_i(C) - \langle C, P \rangle$$

for all  $C \subset \Omega$ .

( $\Leftarrow$ ) Let  $(Y, P)$  be a competitive equilibrium in the economy  $\mathcal{E}$ . Since  $Y$  is a partition,  $Y$  is an allocation in  $\tilde{\mathcal{E}}$ . Because  $Y(i) \in D_i(P)$  for all  $i \in N$ , it follows from (\*) that  $Y(i) \in \tilde{D}_i(P)$  for all  $i \in N$ . Thus  $(Y, P)$  is a competitive equilibrium in the economy  $\tilde{\mathcal{E}}$ .

( $\Rightarrow$ ) Let  $(X, P)$  be a competitive equilibrium in the economy  $\tilde{\mathcal{E}}$ . Then by definition, we have (i)  $P_{\omega_j} = 0$  for all  $\omega_j$  such that  $X(\omega_j) = \omega_j$  and (ii)  $X(i) \in \tilde{D}_i(P)$  for all  $i \in N$ . It follows from (\*) and (ii) that  $X(i) \in D_i(P)$  for all  $i \in N$ . Let  $W_i = \{\omega_j \in \Omega_i : X(\omega_j) = \omega_j\}$  and  $Y(i) = X(i) \cup W_i$ . Clearly  $Y$  is a partition of  $\Omega$ . Since  $X(i) \in D_i(P)$  and  $u_i$  is weakly monotone for all  $i \in N$ , it follows that  $Y(i) \in D_i(P)$  for all  $i \in N$  (note that  $P_w = 0$  for every  $w \in W_i$ ). Hence  $(Y, P)$  is a competitive equilibrium in the economy  $\mathcal{E}$ .

Finally, it follows from Bondareva [3] and Shapley [18] that the core  $C(V)$  is nonempty if and only if the game  $V$  is balanced. This together with Steps 1 and 2 completes the proof. *Q.E.D.*

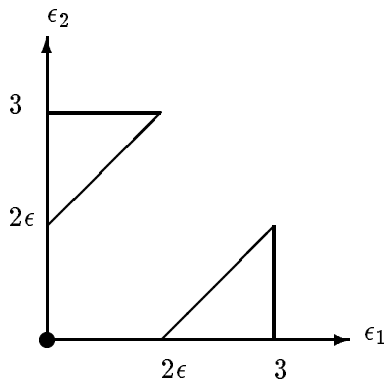
Theorem 1 shows that the balancedness of the game  $V$  does provide a useful necessary and sufficient condition for the existence of competitive equilibrium in the exchange economy  $\mathcal{E}$ . From the construction of the economy  $\tilde{\mathcal{E}}$ , we know precisely what is a competitive price vector. A competitive price vector in the economy  $\mathcal{E}$  is nothing but a core payoff vector received by the commodities “sellers”  $\Omega$  in the game  $V$ .

Next we apply Theorem 1 to study Example 1 to find out which set of parameters admits a competitive equilibrium and which does not. This example may be helpful to show why Theorem 1 is useful. The game  $V$  for Example 1 is defined as follows:

$$\begin{aligned} V(\{j, C\}) &= u_j(C) && \text{for all } C \subset \Omega \\ V(\{k, C\}) &= u_k(C) && \text{for all } C \subset \Omega \\ V(\{C, j, k\}) &= \begin{cases} 4 + \epsilon_2 & C = \{1\} \\ 4 & C = \{2\} \\ 4 + \epsilon_1 & C = \{3\} \\ 8 + \epsilon_2 & C = \{1, 2\} \\ 8 + \epsilon_1 + \epsilon_2 & C = \{1, 3\} \\ 8 + \epsilon_1 & C = \{2, 3\} \end{cases} \end{aligned}$$

$$V(\Omega \cup \{j, k\}) = w(\{i, j, k\})$$

and  $V(C) = 0$  for all  $C \subset \Omega$  or  $C \subset N$ , where  $\Omega = \{1, 2, 3\}$  and  $w(\{i, j, k\})$  is defined as in Example 1. Define  $\mathcal{F} = \{(0, 0) \cup \{(\epsilon_1, \epsilon_2) \in [0, 3] \times [0, 3] : |\epsilon_1 - \epsilon_2| \geq 2\epsilon\}\}$ .



**Figure 1**  $\mathcal{F}$  in Claim (ii).

**Claim (i)** *The game  $V$  above is balanced if and only if*

$$\frac{1}{2}V(\{1, 3, j, k\}) + \frac{1}{2}V(\{1, 2, j\}) + \frac{1}{2}V(\{2, 3, k\}) \leq V(\Omega \cup N);$$

(ii) A competitive equilibrium exists in Example 1 if and only if  $(\epsilon_1, \epsilon_2) \in \mathcal{F}$ .

Claim (ii) is an application of Claim (i) and Theorem 1. The sets of parameters in Example 1 that admit a competitive equilibrium consist of the two triangles and the origin in Figure 1 described by the set  $\mathcal{F}$ . Any other parameters do not have a competitive equilibrium.

Consider the economy  $\mathcal{E}$  in Example 1 with  $\epsilon_1 = \epsilon_2 = \frac{1}{4}$  and  $\epsilon = \frac{1}{2}$ . Kelso and Crawford [11] showed that the constructed economy  $\tilde{\mathcal{E}}$  has an empty core (they showed this by the core conditions). Therefore the economy  $\tilde{\mathcal{E}}$  does not have a competitive equilibrium. Their result together with Step 2 shows that the economy  $\mathcal{E}$  in Example 1 does not have a competitive equilibrium. Alternatively, one can use the condition in Claim (i) to show that when  $\epsilon_1 = \epsilon_2 = \frac{1}{4}$  and  $\epsilon = \frac{1}{2}$ , the game  $V$  is not balanced. Therefore a competitive equilibrium does not exist with these parameters.

**Proof of Claim (i).** Since the game  $V$  is superadditive, Theorem 3 in Shapley [18] shows that the core  $C(V) \neq \emptyset$  iff the game  $V$  is balanced for every proper minimal balanced collection.<sup>6</sup> To find all proper minimal balanced collections, we consider a proper balanced collection  $\mathcal{C} = \{\mathcal{C}_{\{j,k\}}, \mathcal{C}_{\{j\}}, \mathcal{C}_{\{k\}}\}$  of  $\Omega \cup N$  such that each element of  $\mathcal{C}_{\{j,k\}}$ ,  $\mathcal{C}_{\{j\}}$  and  $\mathcal{C}_{\{k\}}$  contains  $\{j, k\}$ ,  $\{j\}$  and  $\{k\}$ , respectively. Let  $x, y$  and  $z$  be the balanced weight of  $\{j, k\}$  of the collection  $\mathcal{C}_{\{j,k\}}$ , the balanced weight of  $\{j\}$  of the collection  $\mathcal{C}_{\{j\}}$  and the balanced weight of  $\{k\}$  of the collection  $\mathcal{C}_{\{k\}}$ , respectively. It follows that  $y = z$  since  $x + y = 1 = x + z$ . Clearly  $x > 0, y > 0$  and  $z > 0$  since  $\mathcal{C}$  is proper. Thus each  $\mathcal{C}_{\{j,k\}}$ ,  $\mathcal{C}_{\{j\}}$  and  $\mathcal{C}_{\{k\}}$  must contain at least one element. Now it follows from Shapley [18] that  $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$  is the unique minimal balanced collection of  $\{1, 2, 3\}$ . Therefore, a minimal balanced collection  $\mathcal{C}$  that is also proper must be one of such one-to-one maps

$$\pi : \{\{j, k\}, \{j\}, \{k\}\} \rightarrow \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

Thus, we must have  $x = y = z = \frac{1}{2}$ . It follows that  $V$  is balanced if and only if

$$\frac{1}{2}V(\{1, 3, j, k\}) + \frac{1}{2}V(\{1, 2, j\}) + \frac{1}{2}V(\{2, 3, k\}) \leq V(\Omega \cup N),$$

since for any coalition  $C \subset \Omega$  such that  $C \neq \Omega$ ,  $V(\{C, j, k\}) \leq V(\{1, 3, j, k\})$ ,  $V(\{C, j\}) \leq V(\{1, 2, j\})$  and  $V(\{C, k\}) \leq V(\{2, 3, k\})$ . Q.E.D.

In Shapley [18] it is shown that the core is nonempty for any (superadditive) three players  $\{1, 2, 3\}$  coalitional form game  $\tilde{w}$  if and only if

$$\frac{1}{2}\tilde{w}(\{1, 3\}) + \frac{1}{2}\tilde{w}(\{1, 2\}) + \frac{1}{2}\tilde{w}(\{2, 3\}) \leq \tilde{w}(\{1, 2, 3\}).$$

Our condition above for the economy in Example 1 is analogous to the result in Shapley [18]. In any general exchange economy with  $N = \{j, k\}$  and  $\Omega = \{1, 2, 3\}$ , it is sufficient and necessary to check six such inequalities (see the map  $\pi$  above) for the existence of competitive equilibrium.

---

<sup>6</sup>A balanced collection is proper if no two elements in it are disjoint (Shapley [18]).

In the close of this section we remark that if agents have multiple units of the same good or different agents have the same good, then Theorem 1 still holds. This is due to two observations. The first one is that multiple copies of the same good will have the same price at any competitive equilibrium. Otherwise there exist arbitrage opportunities at equilibrium. The second one is that each copy, as a commodity seller, of the same good will have the same payoff in all core outcomes of the game  $V$ . This is because all copies of the same good are symmetric in the game  $V$  and any core outcome will assign the same payoff to these symmetric sellers according to the core conditions. Therefore the conditions that identical goods would have to have the same price will be satisfied at all competitive equilibria and in all core outcomes. Also see Bikhchandani and Mamer [1] for a detailed discussion.

## 4 Misrepresentation

Let  $\mathcal{U}^n$  denote the set of all profiles of weakly monotone utility functions such that a competitive equilibrium exists in the economy  $\mathcal{E}(u)$  for every  $u \in \mathcal{U}^n$ . Given an exchange economy  $\mathcal{E}(u)$ , a competitive price vector  $P_*$  is the minimum competitive price vector if it satisfies  $P_* \leq P$  for every competitive price vector  $P$  in the economy. A mechanism  $(\varphi, P)$  is a competitive price mechanism if  $(\varphi(u), P(u))$  is a competitive equilibrium for each profile  $u \in \mathcal{U}^n$  and it is the minimum competitive price mechanism if  $P(u)$  is the minimum competitive price vector for each profile  $u \in \mathcal{U}^n$ . Similarly one may define the maximum competitive price vector and the maximum competitive price mechanism. A mechanism  $(\varphi, P)$  induces a strategic form game and it is individually (coalitionally) strategy proof on the domain  $\mathcal{U}^n$  if it is always the best for every agent (every member in each coalition) to reveal his/her true utility function  $u_i$  for every profile  $u \in \mathcal{U}^n$ .

In the Vickrey auction model and the assignment problem, the minimum competitive price mechanism is individually strategy proof and it is also coalitionally strategy proof if side payments are prohibited. We show that the nice incentive properties of the minimum competitive price mechanism in these two models do not carry over to the current exchange economy. Indeed examples of exchange economies exist for which no competitive price mechanism is individually (coalitionally) strategy proof.

**Proposition 1** *There exists an exchange economy such that no competitive price mechanism is individually (coalitionally) strategy proof.*

**Proof.** Let  $\epsilon_1 = 0$ ,  $\epsilon_2 = 0$  and  $\epsilon = \frac{1}{2}$  in Example 1. Also let  $u = (u_i, u_j, u_k)$  be the underlying true profile of utility functions defined in Example 1 with these parameters. Note that this economy  $\mathcal{E}(u)$  is bilateral in nature as in the assignment problem.

There are two optimal allocations  $Y$  and  $Y'$  in the economy  $\mathcal{E}(u)$  defined by

$$Y(j) = \{1, 2\}, Y'(j) = \{1\} \text{ and } Y(k) = \{3\}, Y'(k) = \{2, 3\}.$$

Let  $P$  be any competitive price vector in the economy  $\mathcal{E}(u)$ . It follows from  $Y(j), Y'(j) \in D_j(P)$  that  $u_j(\{1, 2\}) - P_1 - P_2 = u_j(\{1\}) - P_1$ . Thus  $P_2 = u_j(\{1, 2\}) - u_j(\{1\}) = 3\frac{1}{2}$ . It also follows from  $Y'(j) \in D_j(P)$  that  $u_j(\{1\}) - P_1 \geq u_j(\{1, 3\}) - P_1 - P_3$ . Thus  $P_3 \geq 3$ . By symmetry, we have that  $P_1 \geq 3$  and  $P_1 = P_3$ . We claim that  $P_* = (3, 3\frac{1}{2}, 3)$  is the minimum competitive price vector. To see this, it is sufficient to show that  $P_* = (3, 3\frac{1}{2}, 3)$  is competitive. But this is an easy task to check.

At the minimum competitive price vector  $P_*$ , agents  $j$  and  $k$  each obtain trading profits 1. Since they are buyers not sellers, the minimum competitive price mechanism is the most favorable one to them. Thus the trading profits for agents  $j$  and  $k$  are at most 1 for any competitive price mechanism when they report truthfully.

Given agents  $i$  and  $k$  reporting their truth, we show that agent  $j$  can obtain more than 1 for any competitive price mechanism by misreporting her utility functions as follows:

$$\begin{aligned} \tilde{u}_j(\{1\}) &= \tilde{u}_j(\{2\}) = \tilde{u}_j(\{3\}) = 1\frac{1}{2} \\ \tilde{u}_j(\{1, 2\}) &= \tilde{u}_j(\{1, 3\}) = \tilde{u}_j(\{2, 3\}) = 3, \tilde{u}_j(\{1, 2, 3\}) = 4.5. \end{aligned}$$

Note that  $\tilde{u}_j$  is monotone. Let  $\tilde{u} = (u_i, \tilde{u}_j, u_k)$ . There are two optimal allocations  $X$  and  $X'$  in the economy  $\mathcal{E}(\tilde{u})$  defined by

$$X(j) = \emptyset, X'(j) = \{1\} \text{ and } X(k) = \{1, 2, 3\}, X'(k) = \{2, 3\}.$$

Note that agent  $j$  reduces the competition in the demand side by such misrepresentation.

Let  $Q$  be any competitive price vector in the economy  $\mathcal{E}(\tilde{u})$ . It follows from  $X(k), X'(k) \in D_k(Q)$  that  $Q_1 = 1\frac{1}{2}$ , and

$$Q_1 \leq Q_2 \leq Q_1 + \frac{1}{2} \text{ and } Q_1 \leq Q_3 \leq Q_1 + \frac{1}{2}.$$

Thus agent  $j$  obtains (true) trading profits  $2\frac{1}{2}$  in the economy  $\mathcal{E}(\tilde{u})$  for any competitive price mechanism. But she obtains at most 1 when she reveals her true utility functions for any competitive price mechanism in the economy  $\mathcal{E}(u)$ . This shows that no competitive price mechanism in the economy  $\mathcal{E}(u)$  is individually strategy proof.

We now show that no competitive price mechanism in  $\mathcal{E}(u)$  is also coalitionally strategy proof. Since  $(1\frac{1}{2}, 2, 2)$  is competitive, it must be the maximum competitive price vector in the economy  $\mathcal{E}(\tilde{u})$ , at which agent  $k$  obtains trading profits  $3\frac{1}{2}$ . Since agent  $k$  is a buyer not a seller, he can obtain at least  $3\frac{1}{2}$  for any competitive price mechanism in the economy  $\mathcal{E}(\tilde{u})$ . But agent  $k$  obtains at most 1 in the economy  $\mathcal{E}(u)$  when agent  $j$  reports her true utility functions. Thus agent  $k$  gets

better off when agent  $j$  misreports her utility functions by  $\tilde{u}_j$  for any competitive price mechanism in the economy  $\mathcal{E}(\tilde{u})$ . Thus each member in the coalition  $\{j, k\}$  can do better for any competitive price mechanism when agent  $j$  misreports her utility functions by  $\tilde{u}_j$ , without side payments. This completes the proof. *Q.E.D.*

## References

- [1] Sushil Bikhchandani and John W. Mamer, Competitive Equilibrium in an Exchange Economy with Indivisibilities, Anderson Graduate School of Management, UCLA, 1994. Mimeo. To appear in *Journal of Economic Theory*.
- [2] O.N. Bondareva, Some Applications of Linear Programming Methods to the Theory of Cooperative Games, *Problemy Kybernetiki* **10** (1963), 119-139.
- [3] Gabrielle Demange, Strategyproofness in the Assignment Market Game, Laboratoire d'Econometrie de l'Ecole Polytechnique, Paris, 1982. Mimeo.
- [4] Gabrielle Demange and David Gale, The Strategy Structure of Two-sided Matching Markets, *Econometrica* **53** (1985), 873-88.
- [5] David Gale, Equilibrium in a Discrete Exchange Economy with Money, *Int. J. of Game Theory* **13** (1984), 61-4.
- [6] David Gale and Lloyd Shapley, College Admissions and the Stability of Marriage, *Amer. Math. Monthly* **69** (1962), 9-15.
- [7] Faruk Gül and Ennio Stacchetti, Walrasian Equilibrium without Complementarities, Princeton University, 1996. Mimeo.
- [8] Mamoru Kaneko, The Central Assignment Game and the assignment Markets, *J. of Math. Econ.* **10** (1982), 205-32.
- [9] Mamoru Kaneko, Housing Markets with Indivisibilities, *J. of Urban Econ.* **13** (1983), 22-50.
- [10] Mamoru Kaneko and Yoshitsugu Yamamoto, The Existence and Computation of Competitive Equilibria in Markets with an Indivisible Commodity, *J. of Econ. Theory* **38** (1986), 118-36.
- [11] Alexander S. Kelso, Jr. and Vincent P. Crawford, Job Matching, Coalition Formation, and Gross Substitutes, *Econometrica* **50** (1982), 1483-1504.
- [12] T. Koopmans and M. Beckman, Assignment Problems and the Location of Economic Activities, *Econometrica* **25** (1957), 53-76.
- [13] Herman B. Leonard, Elicitation of Honest Preferences for the Assignment of Individuals to Positions, *J. of Polit. Econ.* **91** (1983), 461-79.
- [14] John McMillan, Selling Spectrum Rights, *The J. of Econ. Perspectives* **8** (1994), 145-162.

- [15] Martine Quinzii, Core and Competitive Equilibria with Indivisibilities, *Int. J. of Game Theory* **13** (1984), 41-60.
- [16] Alvin E. Roth, Stability and Polarization of Interests in Job Matching, *Econometrica* **52** (1984), 47-57.
- [17] Alvin E. Roth and Marilda Sotomayor, Two-sided Matching: a Study in Game-theoretic Modeling and Analysis, Cambridge: Cambridge University Press, 1990.
- [18] Lloyd Shapley, On Balanced Sets and Cores, *Naval Res. Logistics Quarterly* **14** (1967), 453-460.
- [19] Lloyd Shapley and Herbert Scarf, On Core and Indivisibility, *J. of Math. Econ.* **1** (1974), 23-8.
- [20] Lloyd Shapley and Martin Shubik, The Assignment Game I: the core, *Int. J. of Game Theory* **1** (1972), 111-30.
- [21] W. Vickrey, Counterspeculation, Auctions, and Competitive Sealed Tenders, *J. of Finance* **16** (1961), 8-37.